

Holomorphic differentials of solvable Galois towers of curves over a perfect field

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Wednesday 28th December, 2016

Abstract

We give a basis for the space of holomorphic differentials for a natural class of solvable Galois towers of curves with perfect field of constants of characteristic $p > 0$, which depends upon the existence of a global standard form for generators of the tower. We also describe the Galois action on the space of holomorphic differentials when the Galois group is abelian. Finally, we extend a result of Madan and Valentini for a cyclic Galois group when the field of constants is not algebraically closed.

1 Introduction

Let L be a function field of one variable with field of constants k , and let G be any group of automorphisms of L which fix k . We are interested in understanding the structure of the k -vector space Ω_L of holomorphic differentials of L , which forms a $k[G]$ -module. This problem was first introduced by Hurwitz [11]. If the characteristic of k is equal to 0 and k is algebraically closed, then the structure of Ω_L as a $k[G]$ -module was essentially solved by Chevalley and Weil in response to a question of Hecke [5]. If $\text{char } k = p > 0$, this structure is not well understood except when ramification is tame, where methods from characteristic 0 can be adapted [12, 19]. In general, this remains an open problem in the presence of wild ramification [21]. Boseck [3] first studied this problem in positive characteristic for Artin-Schreier and Kummer extensions via an explicit basis according to Hasse's standard form, which always exists over a rational field. Other more recent work addresses cases of restricted ramification or group structure, e.g., if ramification is weak or total, G is cyclic or an unramified p -group, or if L is a cyclotomic function field [13, 23, 24, 27, 31].

The purpose of this paper is to provide a formulation of Boseck's basis and its $k[G]$ -module structure for certain natural solvable towers over a rational function field. We assume that a global standard form exists at each step of the tower (see §2 for a complete description of the conditions needed for this), and that the place infinity in the rational field is unramified in the tower. We note that Boseck himself assumed that this place at infinity is unramified in his results on Artin-Schreier and Kummer extensions of a rational function field (Ibid.). It is difficult to describe in simple terms the class of towers we consider (§2). However, from these towers we are able to recover all previous known results on the construction of Boseck bases over the rational field, as well as much more. Our efforts in this paper are focused on unifying the theory of Boseck bases of Kummer and Artin-Schreier extensions for use in towers, but we would like to point out that Boseck bases also have many uses. For example, Garcia [6, 7]

used bases of this type for elementary abelian extensions to compute Weierstrass points, and Madden [18] used these to calculate the rank of the Hasse-Witt matrix.

For our main result, we construct an explicit Boseck k -basis of Ω_L (Theorem 4.1) for certain towers of Kummer and Artin-Schreier extensions, after unification of the Boseck basis for Kummer and Artin-Schreier extensions (see Lemmas 3.3 and 3.6). We do not require that k be an algebraically closed field, but rather, we need only assume that k is a perfect field (in particular, k may be a finite field). We note that in order to construct the natural Boseck k -basis of Ω_L , the arguments we employ are local and depend on ramification groups. This basis also admits a natural G -action on Ω_L , which we describe in §6. We also give a natural decomposition of Ω_L into $k[G]$ -submodules when G is abelian (Theorem 6.5). In special cases, our basis and the associated G -module action agree with constructions appearing in the work of Garcia and Stichtenoth, Madden, Rzedowski-Calderòn et al., Valentini and Madan, and numerous others [8, 13, 18, 23, 24, 29]. We do not know to what extent such a strikingly simple decomposition is possible for arbitrary curves in characteristic p .

In §5, we provide an algorithm for passing from global standard form for a composita of Kummer and Artin-Schreier extensions of a rational function field to a tower. This alone allows one to construct via Theorem 4.1 the Boseck k -basis of Ω_L for a large class of towers. Difficulties presented by global standard form appear to us to be related to the structure of the class group. As we explain in §3, a global standard form does not in general exist over a non-rational function field. On the other hand, we give several examples (§5.1) over a non-rational field when global standard form does exist.

Finally, we note that when K is not rational, G is cyclic, and k is algebraically closed, the situation is much different: It is then possible to describe the representation of G [13, 29] without identification of an explicit basis using invariants due to Boseck. We are able to complete this proof, so that this result holds over any perfect field of constants k (Lemma 5.5). For this, we find that it suffices to use Riemann-Roch and various techniques in approximation theory (Lemmas 5.1 and 5.3).

2 Notation and assumptions

We let k be a perfect field of characteristic $p > 0$ and L a function field over k . We make the following assumptions. We suppose that:

- The constant field of L is equal to k .
- There is a rational field $K = k(x)$ over k such that L/K is a finite Galois extension, and that L/K may be expressed as a tower of cyclic Galois extensions

$$(1) \quad L = L_r/L_{r-1} \cdots /L_1/L_0 = K,$$

where for each $i = 1, \dots, r$, the extension L_i/L_{i-1} possesses cyclic Galois group of order n_i (thus $n_i \mid [L : K]$), with either $n_i = p$ or n_i coprime to p .

- There is a choice of a generator x for the rational field K such that the place at infinity for x is unramified in L .
- The field of constants k contains the n_i th roots of unity, for any $i = 1, \dots, r$ such that n_i is coprime with p . (This is done simply to ensure that all cyclic extensions in the tower (1) of degree n_i coprime with p are Kummer.)

- For each $i = 1, \dots, r$, it is possible to find a generator in global standard form for L_i/L_{i-1} ; we shall define global standard form later in this section (see Definitions 2.1 and 2.4).
- If L_i/L_{i-1} is Kummer, there does not exist an integer $d > 1$ such that

$$d \mid \gcd(v_{\mathfrak{p}_{i-1}}(c_i), n_i)$$

for all places \mathfrak{p}_{i-1} of L_{i-1} which ramify in L_i . We will see in section 2.2 how this cannot occur when L_{i-1} is a rational field, and how this is related to unramified subextensions of L_i/L_{i-1} .

We shall explain later why these are necessary for the construction of the k -basis of Ω_L . Naturally, we would like to know when one may obtain a Galois tower satisfying all these assumptions. In this article, we do not provide a complete answer to this difficult question; instead, we give some interesting explicit examples where such a tower may be obtained (see §5).

We also prove (§3.1 and 3.2) that the existence of a geometric unramified Galois extension of a function field proves that a global standard form does not always exist, as any unramified Kummer or Artin-Schreier extension in global standard form must be a constant extension. In other words, the requirement of existence of global standard form excludes the possibility of unramified geometric steps in the tower. We emphasise that it is essentially only the choice of L which is fixed: If one is able to find k , K , x , and L_i ($i = 1, \dots, r$) which matches all the above assumptions, then one obtains a basis of holomorphic differentials. Later in the paper, we will see how this gives us some flexibility in the choice of basis in certain cases.

We denote by \mathbb{P}_K the set of all places in K which ramify in L . For simplicity of notation, we henceforth adopt the convention of denoting by \mathfrak{P} a place above \mathcal{P} . Also, for each finite place $\mathcal{P} \in \mathbb{P}_K$, we denote by $d_{\mathcal{P}}$ the degree of the place \mathcal{P} and $p_{\mathcal{P}}(x)$ the irreducible polynomial associated with a place \mathcal{P} , which is a prime ideal of $k[x]$. For each $i = 1, \dots, r$, we let $\mathfrak{p}_i = \mathfrak{P} \cap L_i$ and \mathbb{P}_{L_i} the set of places of L_{i-1} which ramify in L_i .

3 Preliminaries

3.1 Artin-Schreier extensions

If $n_i = p$, then the extension L_i/L_{i-1} is Artin-Schreier, i.e., there exists a primitive element y_i , called an *Artin-Schreier generator*, such that

$$y_i^p - y_i = c_i \in L_{i-1},$$

with $c_i \neq w^p - w$ for all $w \in L_i$ [30, Theorem 5.8.4]. There exists a generator of the Galois group $\text{Gal}(L_i/L_{i-1}) \simeq \mathbb{Z}/p\mathbb{Z}$ which acts on the element y_i via $y_i \rightarrow y_i + 1$. As in (§2), we suppose that places \mathfrak{p}_{i-1} of L_{i-1} above the place \mathcal{P}_{∞} of $L_0 = K$ corresponding to the pole of the element x are unramified in L_i .

Definition 3.1. *We say that an Artin-Schreier generator y_i of L_i/L_{i-1} is in global standard form if, for any choice of place \mathfrak{p}_{i-1} of L_{i-1} , $v_{\mathcal{P},i} := v_{\mathfrak{p}_{i-1}}(c_i) = v_{\mathfrak{p}_i}(y_i) \geq 0$ if \mathfrak{p}_{i-1} is unramified in L_i , and otherwise $v_{\mathcal{P},i} < 0$ and $\gcd(v_{\mathcal{P},i}, p) = 1$.*

As mentioned previously, if L_i/L_{i-1} were an unramified extension, then the existence of a global standard form for L_i/L_{i-1} would imply that $v_{\mathfrak{p}_{i-1}}(c_i) \geq 0$ at any place \mathfrak{p}_{i-1} of L_{i-1} , which

would in turn imply that $c_i \in k$. As k is algebraically closed in L , this would then imply that $y_i \in k$, so that the extension L_i/L_{i-1} is trivial (also, by the Riemann-Hurwitz genus formula, we know that unramified geometric Artin-Schreier extensions do not exist over $k(x)$). In particular, as we suppose the existence of a generator in global standard form for each L_i/L_{i-1} , this implies that none of the Artin-Schreier steps L_i/L_{i-1} ($i = 1, \dots, r$) is unramified. For an Artin-Schreier extension over the rational field $k(x)$, the existence of a generator in global standard form for an Artin-Schreier was proven by Hasse [9].

We note that any generator in global standard form has the same valuation at a given ramified place, and that one can always find such a generator locally in standard form, i.e., at a single choice of place \mathfrak{p}_{i-1} [25, Lemma 3.7.7]. We call this a *local* standard form. It is only when one has an Artin-Schreier generator in a local standard form that one may give a formula for the ramification index and differential exponent at that place in terms of the generator; this applies at one place but does not imply that this choice of generator gives a global standard form. Given a generator in local standard form, the place \mathfrak{p}_{i-1} is ramified (and hence $e_i = p$) if, and only if $v_{\mathcal{P},i} < 0$, and the differential exponent satisfies

$$d(\mathfrak{p}_i|\mathfrak{p}_{i-1}) = (p-1)(1 - v_{\mathcal{P},i}).$$

The ramification filtration $\{G_n(\mathfrak{p}_i)\}_{n=0}^\infty$ has only one jump, occurring at $n = 1 - v_{\mathcal{P},i}$ (Proposition 3.7.8, [25]). We recall here the well known Riemann-Hurwitz formula for geometric Artin-Schreier extensions, which becomes important for construction of the Boseck basis, together with the existence of a Artin-Schreier generator in standard form.

Lemma 3.2. [25, Proposition 3.7.8] *For a geometric Artin-Schreier extension L/K , where K is a function field of genus g_K , the genus of L is given by*

$$g_L = 1 - p + p \cdot g_K + \frac{1}{2} \sum_{\mathcal{P} \in \mathbb{P}_K} d_{\mathcal{P}} \cdot (p-1) \cdot J_{\mathcal{P}},$$

where $J_{\mathcal{P}}$ is the jump of the ramification group filtration at the place \mathcal{P} , equal to $1 - v_{\mathcal{P}}$, and $v_{\mathcal{P}}$ denotes the valuation of the ramified place in standard form.

Note that over a rational field, any Artin-Schreier extension has a Artin-Schreier generator in standard form [30, Example 5.8.8], which permitted Boseck to give an explicit basis for the space of holomorphic differentials of an Artin-Schreier extension of a rational function field.

Lemma 3.3. [3, Satz 15] *Let $K = k(x)$, and let $K(y) = k(x, y)$ be a geometric Artin-Schreier extension of K degree p , defined by the relation*

$$y^p - y = \frac{g(x)}{\prod_{i=1}^r p_i(x)^{v_i}},$$

where for each $i = 1, \dots, r$, $p_i(x)$ and $g(x)$ are relatively prime polynomials in $k[x]$, d_i denotes the degree of the monic irreducible polynomial $p_i(x)$, and $v_i \not\equiv 0 \pmod{p}$. Suppose furthermore that the element x is chosen so that the place at infinity is unramified. The ramified places in $K(y)$ are precisely those of K associated with $p_i(x)$ for each $i = 1, \dots, r$, and these are fully ramified with ramification index p . For each $\mu \in \{0, \dots, p-1\}$, define λ_i^μ and ρ_i^μ according to the following formula:

$$p\lambda_i^\mu + \rho_i^\mu = (p-1-\mu)v_i + p-1,$$

with $0 \leq \rho_i^\mu \leq p-1$. For each such μ , let $g_\mu(x) \in k[x]$ be defined as

$$g_\mu(x) = \prod_{i=1}^r (p_i(x))^{\lambda_i^\mu},$$

and let

$$t^\mu = \sum_{i=1}^r d_i \lambda_i^\mu.$$

Then the set

$$\mathfrak{B}_L = \{x^\nu [g_\mu(x)]^{-1} y^\mu dx \mid 0 \leq \nu \leq t^\mu - 2, 0 \leq \mu \leq p-2\}$$

forms a k -basis of the space of Ω_L of holomorphic differentials of L .

3.2 Kummer extensions

As for each $i \in \{1, \dots, r\}$ with $\gcd(n_i, p) = 1$, the extension L_i/L_{i-1} is Kummer, one may find a primitive element y_i , called a *Kummer generator*, such that $L_i = L_{i-1}(y_i)$, $y_i^{n_i} = c_i \in L_{i-1}$, and $c_i \neq w^v$, for all $v \mid n_i$ and $w \in L_i$. For a given primitive n_i th root of unity ζ , there exists a generator of the Galois group $\text{Gal}(L_i/L_{i-1}) \simeq \mathbb{Z}/n_i\mathbb{Z}$ which acts on the element y_i via $y_i \rightarrow \zeta y_i$. We note that $v_{\mathfrak{p}_{i-1}}(c_i)$ is not divisible by n_i at any place \mathfrak{p}_{i-1} of L_{i-1} ramified in L_i . As in (§2), we suppose that places \mathfrak{p}_{i-1} above the place \mathcal{P}_∞ of $L_0 = K$ corresponding to the pole of the element x are unramified, which is equivalent to $n_i \mid v_{\mathfrak{p}_{i-1}}(c_i)$.

Definition 3.4. We say that a Kummer generator y_i of L_i/L_{i-1} is in global standard form if $0 \leq v_{\mathfrak{p}_{i-1}}(c_i) < n_i$ at all places \mathfrak{p}_{i-1} of L_{i-1} ramified in L_i , and $v_{\mathfrak{p}_{i-1}}(c_i) = 0$ at all places \mathfrak{p}_{i-1} of L_{i-1} unramified in L_i , with the exception of those places \mathfrak{p}_{i-1} of L_{i-1} above the place \mathcal{P}_∞ of $L_0 = K$ corresponding to the pole of the element x for which we suppose that $v_{\mathfrak{p}_{i-1}}(c_i) \leq 0$.

The existence of a generator in global standard form for a Kummer extension over a rational field has been proven by Hasse [9]. Over an arbitrary function field K , we know that a generator in global standard form does not always exist. As evidence, we use the case where K possesses an unramified geometric extension (this implies, in particular, that K is not rational). For such a K and such an extension, let y be a generator such that $y^n = c \in K$. If y were in global standard form, then $v_{\mathcal{P}}(c) = 0$ at all places \mathcal{P} that are not above \mathcal{P}_∞ and $v_{\mathcal{P}}(c) \leq 0$ at the places above infinity, which is impossible unless the extension is constant. As a global standard form does exist over a rational field, this also implies that unramified geometric Kummer extensions do not exist over a rational field, a fact which also follows by Riemann-Hurwitz.

In order to prove Theorem 4.1, it is necessary to assume that there exists no integer $d > 1$ such that

$$d \mid \gcd(v_{\mathfrak{p}_{i-1}}(c_i), n_i)$$

for all places \mathfrak{p}_{i-1} of L_{i-1} which ramify in L_i . This is linked to the existence of unramified geometric subextensions in L_i/L_{i-1} , for which we know that global standard form does not exist. To see this, suppose that $v_{\mathfrak{p}_{i-1}}(c_i) = l_{\mathfrak{p}_{i-1}}$ for any such ramified place and denote $d = \gcd(l_{\mathfrak{p}_{i-1}}, n_i)$ this common factor. With $u = y^{n_i/d}$, the Kummer subextension $L_{i-1}(u)/L_{i-1}$ has $u^d = c_i$, and by assumption, $d \mid v_{\mathfrak{p}_{i-1}}(c_i)$, for any ramified places \mathfrak{p}_{i-1} in L_i/L_{i-1} . As a consequence of this,

$L_{i-1}(u)/L_{i-1}$ is unramified. Boseck did not need to assume this in creating an explicit basis for Kummer extensions over the rational field, as unramified extensions of the rational field simply do not exist. The same is true in the work of Valentini and Madan [29].

It is known that for a fixed choice of place \mathfrak{p}_{i-1} of L_{i-1} , y_i may be chosen in *local* standard form at \mathfrak{p}_{i-1} [30, Theorem 5.8.12], so that $0 = v_{\mathfrak{p}_{i-1}}(c_i)$ if \mathfrak{p}_{i-1} is unramified and $v_{\mathfrak{p}_{i-1}}(c_i) > 0$ if \mathfrak{p}_{i-1} ramified in L_i , where the valuation $v_{\mathfrak{p}_{i-1}}(c_i)$ of c_i is viewed in the field L_{i-1} via $y_i^{n_i} = c_i$. For each place \mathfrak{p}_i of L_i ramified above L_{i-1} , the ramification index in L_i/L_{i-1} satisfies

$$e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) = \frac{n_i}{\gcd(n_i, v_{\mathfrak{p}_{i-1}}(c_i))},$$

where the valuation $v_{\mathfrak{p}_{i-1}}$ is in local standard form, and the differential exponent is equal to

$$d(\mathfrak{p}_i|\mathfrak{p}_{i-1}) = \frac{n_i}{\gcd(n_i, v_{\mathfrak{p}_{i-1}}(c_i))} - 1.$$

We note furthermore that $v_{\mathcal{P},i} := v_{\mathfrak{p}_i}(y_i) = e(\mathfrak{p}_i|\mathfrak{p}_{i-1})v_{\mathfrak{p}_{i-1}}(c_i)/n_i$ is coprime with n , for any place \mathfrak{p}_i of L_i above a ramified place \mathfrak{p}_{i-1} of L_{i-1} . A Kummer extension is of degree coprime to p , thus tamely ramified, and as a consequence the ramification filtration $\{G_n(\mathfrak{p}_i)\}_{n=0}^\infty$ at \mathfrak{p}_i has only one jump, which occurs at $n = 1$ [25, Proposition 3.7.3]. We recall again the well-known genus formula for geometric Kummer extensions (ibid.).

Lemma 3.5. *For a geometric Kummer extension L/K of degree n , where K is a function field of genus g_K , the genus of L is given by*

$$g_L = 1 - n + n \cdot g_K + \frac{1}{2} \sum_{\mathcal{P} \in \mathbb{P}_K} (e(\mathfrak{P}|\mathcal{P}) - 1) \cdot \frac{n}{e(\mathfrak{P}|\mathcal{P})} \cdot d_{\mathcal{P}}.$$

It is important to note that, unlike in Artin-Schreier extensions, the term $n/e(\mathfrak{P}|\mathcal{P})$ appears in the Riemann-Hurwitz formula, as partial ramification is possible in Kummer extensions. In order to construct a k -basis of Ω_L using local ramification data, we must transform Boseck's formulae for Kummer and Artin-Schreier extensions to obtain a type of basis which is consistent across such extensions. The primary obstruction to a direct application of the Boseck basis is the sign difference in the power of the generator in the Kummer versus the Artin-Schreier case: For Kummer extensions, the power of a generator appearing in the Boseck basis is negative, whereas it is positive for Artin-Schreier extensions. We therefore construct an alternate form of the Boseck basis for Kummer extensions in the following lemma.

Lemma 3.6. *Let $K = k(x)$, and let $K(y) = k(x, y)$ be a geometric Kummer extension of K degree n , defined by the relation*

$$y^n = f(x) \in k[x]$$

in global standard form with

$$f(x) = \alpha \cdot \prod_{i=1}^r (p_i(x))^{v_i},$$

where for each $i = 1, \dots, r$, d_i denotes the degree of the monic irreducible polynomial $p_i(x)$ and $0 < v_i$. Suppose that the place of K at infinity, corresponding to the pole of x , is unramified

in $K(y)$ (which is equivalent to requiring $n \mid \deg f(x)$). Let $v = \sum_{i=1}^r v_i d_i$ denote the degree of $f(x)$, and for each $i = 1, \dots, r$, let $m_i = \frac{e_i v_i}{n}$, where e_i is the ramification index for $p_i(x)$ in $K(y)$. For each $\mu \in \{1, \dots, n-1\}$, we define λ_i^μ and ρ_i^μ according to the following formula:

$$e_i \lambda_i^\mu + \rho_i^\mu = \mu m_i + e_i - 1,$$

with $0 \leq \rho_i^\mu \leq e_i - 1$. For each such μ , let $g_\mu(x) \in k[x]$ be defined as

$$g_\mu(x) = \prod_{i=1}^r (p_i(x))^{\lambda_i^\mu},$$

and let

$$t^\mu = \sum_{i=1}^r \frac{d_i}{e_i} (e_i - 1 - \rho_i^\mu).$$

Then the set

$$\mathfrak{B}_{K(y)} = \{x^\nu [g_\mu(x)]^{-1} y^\mu dx \mid 0 \leq \nu \leq t^\mu - 2, 1 \leq \mu \leq n-1\}$$

forms a k -basis of the space of $\Omega_{K(y)}$ of holomorphic differentials of $K(y)$.

Proof. The argument follows similarly to the proof of Satz 16 of [3]. For each $i = 1, \dots, r$, m_i is equal to the valuation of y at a ramified place of $K(y)/K$. We thus find that the divisor of the differential $[g_\mu(x)]^{-1} y^\mu dx$ in $K(y)$ is equal to

$$\begin{aligned} ([g_\mu(x)]^{-1} y^\mu dx)_{K(y)} &= \prod_{i=1}^r \mathfrak{P}_i^{e_i - 1 - e_i \lambda_i^\mu + \mu m_i} \cdot (\text{Con}_{K/K(y)}(\mathcal{P}_\infty))^{\sum_{i=1}^r d_i \lambda_i^\mu - \mu (\frac{1}{n} \sum_{i=1}^r v_i d_i) - 2} \\ &= \prod_{i=1}^r \mathfrak{P}_i^{e_i - 1 - e_i \lambda_i^\mu + \mu m_i} \cdot (\text{Con}_{K/K(y)}(\mathcal{P}_\infty))^{\frac{1}{n} \sum_{i=1}^r d_i \frac{\mu}{e_i} (e_i \lambda_i^\mu - \mu m_i) - 2} \\ &= \prod_{i=1}^r \mathfrak{P}_i^{\rho_i^\mu} \cdot (\text{Con}_{K/K(y)}(\mathcal{P}_\infty))^{\sum_{i=1}^r d_i (e_i - 1 - \rho_i^\mu) / e_i - 2}, \end{aligned}$$

where $\text{Con}_{K/K(y)}$ denotes the conorm of ideals of K into $K(y)$. By definition of t^μ , it follows that the differential $[g_\mu(x)]^{-1} y^\mu dx$ is holomorphic provided that $t^\mu \geq 2$. By the requirement $\mu \in \{1, \dots, n-1\}$, it follows that $t^\mu \geq 1$. To see this, notice that by definition, $\rho_i^\mu \leq e_i - 1$, and thus t^μ is always nonnegative (and an integer; see [3, Satz 16]). If $t^\mu < 1$, then $t^\mu = 0$, which then implies that $\rho_i^\mu = e_i - 1$ for all $i = 1, \dots, r$. It follows for each $i = 1, \dots, r$ that

$$e_i \lambda_i^\mu = \mu m_i = \frac{\mu e_i v_i}{n}.$$

In particular, we obtain that $n \mid \mu v_i$. As $\mu < n$, it follows that there is a factor d of n ($d > 1$) which divides v_i , for all $i = 1, \dots, r$. We thus obtain $y^n = z^d$ with $z \in K$. Let $u = (y^{n/d}/z) \in K$, then $u^d = 1$. As k contains the d th roots of unity, this contradicts that the degree of $K(y)/K$ is equal to n . Also, by definition of \mathfrak{B}_L , there exist no holomorphic differentials with $t^\mu = 1$. Therefore, the number of such differentials which are holomorphic is equal to

$$(2) \quad \sum_{\mu=1}^{n-1} (t^\mu - 1),$$

as $t^\mu = 0$ cannot occur as mentioned previously and $t^\mu = 1$ does not contribute to this sum. Therefore, by Riemann-Hurwitz [30, Corollary 9.4.3], the quantity (2) is equal to

$$\begin{aligned}
\sum_{\mu=1}^{n-1} (t^\mu - 1) &= \sum_{\mu=1}^{n-1} \left[\left(\sum_{i=1}^r \frac{d_i}{e_i} (e_i - 1 - \rho_i^\mu) \right) - 1 \right] \\
&= \sum_{i=1}^r d_i \frac{n}{e_i} \left(\frac{e_i - 1}{2} \right) - (n - 1) \\
&= -1 + n(g_K - 1) + \frac{1}{2} \sum_{i=1}^r d_i \frac{n}{e_i} (e_i - 1) \\
&= g_{K(y)},
\end{aligned}$$

where the second equality above is justified by

$$\begin{aligned}
\sum_{\mu=1}^{n-1} (e_i - 1 - \rho_i^\mu) &= \sum_{\mu=0}^{n-1} (e_i - 1 - \rho_i^\mu) \\
&= \sum_{k=1}^{n/e_i} \sum_{\mu=1+(k-1)e_i}^{ke_i-1} (e_i - 1 - \rho_i^\mu) \\
&= \sum_{k=1}^{n/e_i} \sum_{\mu=1+(k-1)e_i}^{ke_i-1} \rho_i^\mu \\
&= \sum_{k=1}^{n/e_i} \frac{e_i(e_i - 1)}{2} \\
&= n \frac{(e_i - 1)}{2}.
\end{aligned}$$

This follows from the identity

$$\sum_{\mu=1+(k-1)e_i}^{ke_i-1} (e_i - 1 - \rho_i^\mu) = \sum_{\mu=1+(k-1)e_i}^{ke_i-1} \rho_i^\mu = \frac{e_i(e_i - 1)}{2},$$

which holds as $\gcd(m_i, n) = 1$, so that the quantities

$$\mu m_i + e_i - 1 \quad (\mu = 1 + (k-1)e_i, \dots, ke_i - 1)$$

form a complete set of residues modulo e_i . It follows that the elements of $\mathfrak{B}_{K(y)}$ form a k -basis of $\Omega_{K(y)}$. \square

As in Lemma 3.3, the k -basis of Ω_L in the case that L/K is an Artin-Schreier extension and $K = k(x)$ is the rational function field also consists of elements of the form $x^\nu [g_\mu(x)]^{-1} y^\mu dx$. This gives a unified expression for the Boseck basis for both Artin-Schreier and Kummer extensions, which in the sequel will allow us to generate the basis for the “mixed” tower (1).

3.3 The Riemann-Hurwitz formula for towers

By previous arguments, for both Artin-Schreier and Kummer extensions, the differential exponent at any place $\mathfrak{p}_{i-1} \in \mathbb{P}_{L_i}$ is given by

$$d(\mathfrak{p}_i|\mathfrak{p}_{i-1}) = (e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1)J_{\mathcal{P},i},$$

where $J_{\mathcal{P},i}$ is the unique jump of the ramification filtration for \mathfrak{p}_i in L_i/L_{i-1} . By Riemann-Hurwitz, we thus obtain the following genus formula in either situation.

Lemma 3.7. *For the extension L_i/L_{i-1} , the genus formula is given by*

$$g_{L_i} = 1 - n_i + n_i \cdot g_{L_{i-1}} + \frac{1}{2} \sum_{\mathfrak{p}_{i-1} \in \mathbb{P}_{L_i}} \frac{n_i}{e(\mathfrak{p}_i|\mathfrak{p}_{i-1})} \cdot (e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1) \cdot J_{\mathcal{P},i} \cdot d_{\mathfrak{p}_{i-1}}.$$

The genus formula for L_i/L_{i-1} ($i = 1, \dots, r$) allows us to obtain a concise genus formula for the Galois tower L/K . The following result accomplishes exactly this.

Lemma 3.8. (i) *The differential exponent $d(\mathfrak{P}|\mathcal{P})$ of $\mathfrak{P}|\mathcal{P}$ in L/K is given by*

$$d(\mathfrak{P}|\mathcal{P}) = \sum_{i \in R_{\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) \cdot (e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1) \cdot J_{\mathcal{P},i},$$

where $R_{\mathcal{P}} \subset \{0, 1, \dots, r-1\}$ denotes the set of indices such that the place \mathfrak{p}_{i-1} is ramified in L_i/L_{i-1} .

(ii) *The Riemann-Hurwitz formula for L/K may be written as*

$$g_L = 1 - [L : K] + \frac{1}{2} \sum_{\mathcal{P} \in \mathbb{P}_K} \frac{[L:K]}{e(\mathfrak{P}|\mathcal{P})} \cdot d_{\mathcal{P}} \cdot \left[\sum_{i \in R_{\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) \cdot (e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1) \cdot J_{\mathcal{P},i} \right]$$

Proof. For all $i = 1, \dots, r$, we have the ramification formula

$$e(\mathfrak{p}_i|\mathcal{P}) = e(\mathfrak{p}_i|\mathfrak{p}_{i-1})e(\mathfrak{p}_{i-1}|\mathcal{P})$$

and differential exponent

$$d(\mathfrak{p}_i|\mathcal{P}) = e(\mathfrak{p}_i|\mathfrak{p}_{i-1})d(\mathfrak{p}_{i-1}|\mathcal{P}) + d(\mathfrak{p}_i|\mathfrak{p}_{i-1}).$$

From previous observations, have $d(\mathfrak{p}_i|\mathfrak{p}_{i-1}) = (e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1) \cdot J_{\mathcal{P},i}$ for each $i = 1, \dots, r$. Thus, the formula for the differential exponent of $\mathfrak{P}|\mathcal{P}$ may be expressed as

$$d(\mathfrak{P}|\mathcal{P}) = \sum_{i \in R_{\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i)(e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1) \cdot J_{\mathcal{P},i},$$

proving (i). For (ii), by definition, the different $\mathcal{D}_{L/K}$ of L over K is equal to

$$\mathcal{D}_{L/K} = \prod_{\mathcal{P} \in \mathbb{P}_K} \mathcal{P} r_{\mathcal{P}}^{d(\mathfrak{P}|\mathcal{P})},$$

where $\mathcal{P} r_{\mathcal{P}}$ denotes the product of all places of L above \mathcal{P} . As L/K is Galois, the inertia degree and ramification index at a place \mathfrak{P} of L above $\mathcal{P} \in \mathbb{P}_K$ is independent of the choice of \mathfrak{P} . Hence, the product of the inertia degree of $\mathfrak{P}|\mathcal{P}$ with the number of places of L above \mathcal{P} is

equal to $[L : K]/e(\mathfrak{P}|\mathcal{P})$ [30, Corollary 5.2.23]. Furthermore, the differential exponent in L at a place $\mathfrak{P}|\mathcal{P}$ is also independent of the choice of \mathfrak{P} . With the help of [25, Corollary 3.1.14], the Riemann-Hurwitz formula for L/K may thus be written [30, Corollary 9.4.3] as

$$g_L = 1 - [L : K] + \frac{1}{2} \sum_{\mathcal{P} \in \mathbb{P}_K} \frac{[L : K]}{e(\mathfrak{P}|\mathcal{P})} \cdot d_{\mathcal{P}} \cdot \left[\sum_{i \in R_{\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i)(e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1) \cdot J_{\mathcal{P},i} \right],$$

as desired. \square

This is a convenient formula for the genus, as it is expressed only in terms of the valuations of global standard form generators and ramification data for the tower. Lemma 3.8 remains valid if L/K is separable but not Galois, provided ramification indices, inertia degrees, and differential exponents are equal for all places of L above a given place of K , for all places of K which ramify in L .

4 Basis of holomorphic differentials

In this section, we provide an explicit description of the k -basis of Ω_L , which is our main theorem. This is done strictly in terms of the ramification data and valuations of global standard form generators of the tower L/K . Our construction additionally requires the modified Boseck basis for Kummer extensions introduced in Lemma 3.5, which allows the tower to consist of steps of both Artin-Schreier and Kummer extensions. We assume all of the previous notation for the extension L/K , as in §2 and 3.

Theorem 4.1. *Let L/K be a geometric Galois extension of degree n , with perfect constant field k of characteristic $p > 0$ and $K = k(x)$ a rational function field, where L/K admits the structure given in (1) and satisfies all of the assumptions outlined in §2. For each $i = 1, \dots, r$, we suppose that each y_i is either a Kummer or Artin-Schreier generator in global standard form. Given a place $\mathcal{P} \in \mathbb{P}_K$, let $R_{\mathcal{P}} = \{i \in \{1, \dots, r\}, \mathfrak{p}_{i-1} \in \mathbb{P}_{L_i}\}$ the set of indices $i = 1, \dots, r$ such that \mathfrak{p}_{i-1} ramifies in L_i/L_{i-1} , let*

$$R_{p,\mathcal{P}} = \{i \in \{1, \dots, r\} | n_i = p, \mathfrak{p}_{i-1} \in \mathbb{P}_{L_i}\}$$

denote the p -subset of $R_{\mathcal{P}}$, i.e., those indices such that $n_i = p$, and let

$$R_{o,\mathcal{P}} = \{i \in \{1, \dots, r\} | n_i \neq p, \mathfrak{p}_{i-1} \in \mathbb{P}_{L_i}\}$$

denote the prime-to- p subset of $R_{\mathcal{P}}$, i.e., those such that $n_i \neq p$. We denote $e_{\mathcal{P}} := e(\mathfrak{P}|\mathcal{P})$ and $v_{\mathcal{P},i} := v_{\mathfrak{p}_i}(y_i)$, where the valuation of y_i is viewed as existing in L_i . For each $i = 1, \dots, r$ and μ_i , let $\Delta_{\mathfrak{p}_i}^{\mu_i}$ be defined according to the following formula:

$$(3) \quad \Delta_{\mathfrak{p}_i}^{\mu_i} = \begin{cases} (p-1-\mu_i) \cdot (-v_{\mathcal{P},i}) + (p-1) & \text{if } i \in R_{p,\mathcal{P}} \\ \mu_i v_{\mathcal{P},i} + (e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1) & \text{if } i \in R_{o,\mathcal{P}} \\ 0 & \text{otherwise.} \end{cases}$$

Set $\mu = (\mu_1, \dots, \mu_r)$. Let $\lambda_{\mathcal{P}}^{\mu}$ and $\rho_{\mathcal{P}}^{\mu}$ be defined by the equation

$$(4) \quad e_{\mathcal{P}} \lambda_{\mathcal{P}}^{\mu} + \rho_{\mathcal{P}}^{\mu} = \sum_{i=1}^r e(\mathfrak{P}|\mathfrak{p}_i) \Delta_{\mathfrak{p}_i}^{\mu_i}, \quad 0 \leq \rho_{\mathcal{P}}^{\mu} \leq e_{\mathcal{P}} - 1.$$

Also, let

$$g_\mu(x) = \prod_{\mathcal{P} \in \mathbb{P}_K} (p_{\mathcal{P}}(x))^{\lambda_{\mathcal{P}}^\mu}.$$

Let $y^\mu = \prod_{j=1}^r y_j^{\mu_j}$ and

$$t^\mu = \sum_{\mathcal{P} \in \mathbb{P}_K} d_{\mathcal{P}} \left(\lambda_{\mathcal{P}}^\mu - \sum_{i \in R_0, \mathcal{P}} \frac{e(\mathfrak{P}|\mathfrak{p}_i)}{e_{\mathcal{P}}} v_{\mathcal{P},i} \mu_i \right),$$

where $d_{\mathcal{P}}$ denotes the degree of the place $\mathcal{P} \in \mathbb{P}_K$ (§2). Define $\Gamma := \prod_{i=1}^r \{0, \dots, n_i - 1\} - \mu^0$, where $\mu^0 = (\mu_1^0, \dots, \mu_r^0)$ with $\mu_i^0 = 0$ if $n_i \neq p$, and $\mu_i^0 = n_i - 1 = p - 1$ otherwise. Then the set

$$\mathfrak{B}_L := \{x^\nu [g_\mu(x)]^{-1} y^\mu dx \mid 0 \leq \nu \leq t^\mu - 2, \mu = (\mu_1, \dots, \mu_r) \in \Gamma\}$$

forms a k -basis of Ω_L .

Proof. The divisor of y^μ in L is given by

$$(y^\mu)_L = \mathfrak{A}_y^\mu \cdot \prod_{\mathcal{P} \in \mathbb{P}_K} \mathcal{P} r^{\sum_{i \in R_0, \mathcal{P}} \mu_i e(\mathfrak{P}|\mathfrak{p}_i) v_{\mathcal{P},i}},$$

for some integral divisor \mathfrak{A}_y^μ of L , where $\mathcal{P}r$ is the product of the places above \mathcal{P} in L . It follows that the divisor in L of the differential $y^\mu dx$ is given by

$$(y^\mu dx)_L = \mathfrak{A}_y^\mu \cdot \prod_{\mathcal{P} \in \mathbb{P}_K} \mathcal{P} r^{\sum_{i \in R_0, \mathcal{P}} e(\mathfrak{P}|\mathfrak{p}_i) \Delta_{\mathfrak{p}_i}^{\mu_i}} (\text{Con}_{K/L}(\mathcal{P}_\infty))^{-2}.$$

As the quantities $\lambda_{\mathcal{P}}^\mu$ and $\rho_{\mathcal{P}}^\mu$ are defined according to (4), multiplication of $y^\mu dx$ by $[g_\mu(x)]^{-1} = \prod_{\mathcal{P} \in \mathbb{P}_K} (p_{\mathcal{P}}(x))^{-\lambda_{\mathcal{P}}^\mu}$ yields the following divisor in L :

$$([g_\mu(x)]^{-1} y^\mu dx)_L = \mathfrak{A}_y^\mu \cdot \prod_{\mathcal{P} \in \mathbb{P}_K} \mathcal{P} r^{\rho_{\mathcal{P}}^\mu} \cdot (\text{Con}_{K/L}(\mathcal{P}_\infty))^{\sum_{\mathcal{P} \in \mathbb{P}_K} d_{\mathcal{P}} \left(\lambda_{\mathcal{P}}^\mu - \sum_{i \in R_0, \mathcal{P}} \frac{e(\mathfrak{P}|\mathfrak{p}_i)}{e_{\mathcal{P}}} v_{\mathcal{P},i} \mu_i \right) - 2}.$$

Thus, the differential $[g_\mu(x)]^{-1} y^\mu dx$ is holomorphic if, and only if,

$$t^\mu := \sum_{\mathcal{P} \in \mathbb{P}_K} d_{\mathcal{P}} \left(\lambda_{\mathcal{P}}^\mu - \sum_{i \in R_0, \mathcal{P}} \frac{e(\mathfrak{P}|\mathfrak{p}_i)}{e_{\mathcal{P}}} v_{\mathcal{P},i} \mu_i \right) \geq 2.$$

Therefore, as the set $\{x^\nu y^\mu \mid 0 \leq \nu \leq t^\mu - 2, \mu \in \Gamma\}$ is linearly independent over k , the k -linearly independent set

$$\mathfrak{B}_L = \{x^\nu [g_\mu(x)]^{-1} y^\mu dx \mid 0 \leq \nu \leq t^\mu - 2, \mu = (\mu_1, \dots, \mu_r) \in \Gamma\}$$

consists solely of holomorphic differentials. Furthermore, we have that $t^\mu \geq 1$ for all $\mu \in \Gamma$: By construction, the integer

$$\lambda_{\mathcal{P}}^\mu - \sum_{i \in R_0, \mathcal{P}} \frac{e(\mathfrak{P}|\mathfrak{p}_i)}{e_{\mathcal{P}}} v_{\mathcal{P},i} \mu_i = \frac{1}{e_{\mathcal{P}}} \left(e_{\mathcal{P}} - 1 - \rho_{\mathcal{P}}^\mu + \sum_{i \in R_{p, \mathcal{P}}} (p - 1 - \mu_i) \cdot (-v_{\mathcal{P},i}) \right) \geq 0.$$

Also by construction, $e_{\mathcal{P}} - 1 - \rho_{\mathcal{P}}^{\mu} \geq 0$ and $(p - 1 - \mu_i) \cdot (-v_{\mathcal{P},i}) \geq 0$. It follows that $t^{\mu} = 0$ if and only if $(p - 1 - \mu_i) \cdot (-v_{\mathcal{P},i}) = 0$, i.e., $\mu_i = p - 1$, for all $i \in R_{p,\mathcal{P}}$, and $\rho_{\mathcal{P}}^{\mu} = e_{\mathcal{P}} - 1$, for any $\mathcal{P} \in \mathbb{P}_K$. For any elements

$$s = (s_i)_{i \in R_{0,\mathcal{P}}} \in S := \prod_{i \in R_{0,\mathcal{P}}} \{0, \dots, n_i/e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1\},$$

we consider the set

$$\Gamma_{s,\mathcal{P}} := \prod_{i \in R_{p,\mathcal{P}}} \{0, \dots, n_i - 1\} \prod_{i \in R_{0,\mathcal{P}}} \{s_i e(\mathfrak{p}_i|\mathfrak{p}_{i-1}), \dots, (s_i + 1)e(\mathfrak{p}_i|\mathfrak{p}_{i-1}) - 1\}.$$

As the elements $v_{\mathcal{P},i} = e(\mathfrak{p}_i|\mathfrak{p}_{i-1})v_{\mathfrak{p}_{i-1}}(c_i)/n_i$ are coprime with n_i for each $i = 1, \dots, r$, $v_{\mathcal{P},i} > 0$ for each $i = 1, \dots, r$ such that L_i/L_{i-1} is Kummer, and $-v_{\mathcal{P},i} > 0$ for each $i = 1, \dots, r$ such that L_i/L_{i-1} is Artin-Schreier, at any ramified place \mathfrak{p}_{i-1} of L_i/L_{i-1} , it follows that for each $s \in S$, the elements $\rho_{\mathcal{P}}^{\mu}$ form a complete set of residues modulo $e_{\mathcal{P}}$ as μ runs through all possible values in the set $\Gamma_{s,\mathcal{P}}$. To see this, we have the identity

$$\rho_{\mathcal{P}}^{\mu} = e_{\mathcal{P}} \left\langle \frac{\sum_{i=1}^r e(\mathfrak{P}|\mathfrak{p}_i) \Delta_{\mathfrak{p}_i}^{\mu_i}}{e_{\mathcal{P}}} \right\rangle,$$

where for an element $x \in \mathbb{R}$, $\langle x \rangle$ denotes the fractional part of x . Also, by construction, the set

$$\left\{ \sum_{i=1}^r e(\mathfrak{P}|\mathfrak{p}_i) \Delta_{\mathfrak{p}_i}^{\mu_i} \right\}_{\mu \in \Gamma_{s,\mathcal{P}}}$$

forms a complete set of residues modulo $e_{\mathcal{P}}$. Therefore, the remainder $\rho_{\mathcal{P}}^{\mu}$ assumes the value $e_{\mathcal{P}} - 1$ exactly $|S| = \prod_{i \in R_{0,\mathcal{P}}} n_i/e(\mathfrak{p}_i|\mathfrak{p}_{i-1})$ times, and this occurs precisely when the values of μ_i are multiples of $e(\mathfrak{p}_i|\mathfrak{p}_{i-1})$, for all $i \in R_{0,\mathcal{P}}$. The number of instances where this occurs is equal to $|S|$, which subsumes all possible values of μ for which $e_{\mathcal{P}} - 1 = \rho_{\mathcal{P}}^{\mu}$.

As argued in the proof of Lemma 3.6, in order to have $t^{\mu} = 0$, it is necessary that $\rho_{\mathcal{P}}^{\mu} = e_{\mathcal{P}} - 1$, for any $\mathcal{P} \in \mathbb{P}_K$. By assumption, for any L_i/L_{i-1} which are Kummer, not all ramified valuations of c_i share a prime factor with n_i , whence $t^{\mu} = 0$ occurs only when $\mu_i = 0$ for all $i \in R_{0,\mathcal{P}}$ and $\mu_i = p - 1$ for all $i \in R_{p,\mathcal{P}}$. Furthermore, for any μ so that $t^{\mu} = 1$, there exist no holomorphic differentials of the form prescribed in the definition of \mathfrak{B}_L .

By the previous argument, we have

$$|\mathfrak{B}_L| = \sum_{\mu \in \Gamma} (t^{\mu} - 1).$$

We must now show that this quantity is equal to the genus g_L of L . By definition of $\Delta_{\mathcal{P}}^{\mu_i}$ (3), we have

$$\sum_{\mu \in \Gamma} \lambda_{\mathcal{P}}^{\mu} = \sum_{\mu \in \Gamma} \left\lfloor \frac{\sum_{i=1}^r e(\mathfrak{P}|\mathfrak{p}_i) \Delta_{\mathfrak{p}_i}^{\mu_i}}{e_{\mathcal{P}}} \right\rfloor$$

$$= \sum_{\mu \in \Gamma} \left(\left(\frac{\sum_{i=1}^r e(\mathfrak{P}|\mathfrak{p}_i) \Delta_{\mathfrak{p}_i}^{\mu_i}}{e_{\mathcal{P}}} \right) - \left\langle \frac{\sum_{i=1}^r e(\mathfrak{P}|\mathfrak{p}_i) \Delta_{\mathfrak{p}_i}^{\mu_i}}{e_{\mathcal{P}}} \right\rangle \right).$$

Via the change of index $\mu_i \rightarrow p - 1 - \mu_i$ for all $i \in R_{p,\mathcal{P}}$ (see also [3, Satz 15]), which does not alter the value of the sum, we may write

$$\begin{aligned} \sum_{\mu \in \Gamma} \lambda_{\mathcal{P}}^{\mu} = & \sum_{\mu \in \Gamma} \left[\left(\frac{\left[\sum_{i \in R_{0,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) \mu_i v_{\mathcal{P},i} - \sum_{i \in R_{p,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) \mu_i v_{\mathcal{P},i} \right] + e_{\mathcal{P}} - 1}{e_{\mathcal{P}}} \right) \right. \\ & \left. - \left\langle \frac{\left[\sum_{i \in R_{0,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) \mu_i v_{\mathcal{P},i} - \sum_{i \in R_{p,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) \mu_i v_{\mathcal{P},i} \right] + e_{\mathcal{P}} - 1}{e_{\mathcal{P}}} \right\rangle \right]. \end{aligned}$$

By construction, for each $s \in S$, the set

$$\left\{ \left[\sum_{i \in R_{0,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) \mu_i v_{\mathcal{P},i} - \sum_{i \in R_{p,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) \mu_i v_{\mathcal{P},i} \right] + e_{\mathcal{P}} - 1 \mid (\mu_1, \dots, \mu_r) \in \Gamma_{s,\mathcal{P}} \right\}$$

forms a complete system of residues modulo $e_{\mathcal{P}}$. As $|S| = \prod_{i \in R_{0,\mathcal{P}}} n_i / e(\mathfrak{p}_i | \mathfrak{p}_{i-1})$, we therefore find that

$$\begin{aligned} \sum_{\mu \in \Gamma} \left(\lambda_{\mathcal{P}}^{\mu} - \sum_{i \in R_{0,\mathcal{P}}} \frac{e(\mathfrak{P}|\mathfrak{p}_i)}{e_{\mathcal{P}}} v_{\mathcal{P},i} \mu_i \right) &= \sum_{\mu \in \Gamma} \left[\frac{- \sum_{i \in R_{p,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) v_{\mathcal{P},i} \mu_i}{e_{\mathcal{P}}} \right] + \frac{[L : K]}{e_{\mathcal{P}}} \cdot \frac{e_{\mathcal{P}} - 1}{2} \\ &= \frac{[L : K]}{e_{\mathcal{P}}} \cdot \frac{1}{2} \left(\left[- \sum_{i \in R_{p,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) v_{\mathcal{P},i} (p - 1) \right] + e_{\mathcal{P}} - 1 \right). \end{aligned}$$

We note that

$$e_{\mathcal{P}} - 1 = \sum_{i=1}^r e(\mathfrak{P}|\mathfrak{p}_i) (e(\mathfrak{p}_i | \mathfrak{p}_{i-1}) - 1) = \sum_{i \in R_p} e(\mathfrak{P}|\mathfrak{p}_i) (e(\mathfrak{p}_i | \mathfrak{p}_{i-1}) - 1).$$

By previous observations, for each $i \in R_{p,p}$, we have $J_{\mathcal{P},i} = 1 - v_{\mathcal{P},i}$, and for each $i \in R_{0,\mathcal{P}}$, we have $J_{\mathcal{P},i} = 1$. Thus, via Lemma 3.8, we obtain

$$\begin{aligned} \sum_{\mu \in \Gamma} t^{\mu} &= \sum_{\mu \in \Gamma} \left(\sum_{\mathcal{P} \in \mathbb{P}_K} d_{\mathcal{P}} \left(\lambda_{\mathcal{P}}^{\mu} - \sum_{i \in R_{0,\mathcal{P}}} \frac{e(\mathfrak{P}|\mathfrak{p}_i)}{e_{\mathcal{P}}} v_{\mathcal{P},i} \mu_i \right) \right) \\ &= \sum_{\mathcal{P} \in \mathbb{P}_K} d_{\mathcal{P}} \cdot \frac{[L : K]}{e_{\mathcal{P}}} \cdot \frac{1}{2} \left(\left[- \sum_{i \in R_{p,\mathcal{P}}} e(\mathfrak{P}|\mathfrak{p}_i) v_{\mathcal{P},i} (p - 1) \right] + e_{\mathcal{P}} - 1 \right) \\ &= \frac{1}{2} \sum_{\mathcal{P} \in \mathbb{P}_K} \frac{[L : K]}{e_{\mathcal{P}}} \cdot d_{\mathcal{P}} \cdot \left[\sum_{i \in R_p} e(\mathfrak{P}|\mathfrak{p}_i) \cdot (e(\mathfrak{p}_i | \mathfrak{p}_{i-1}) - 1) \cdot J_{\mathcal{P},i} \right] \end{aligned}$$

$$= \frac{1}{2} \sum_{\mathcal{P} \in \mathbb{P}_K} \frac{[L : K]}{e_{\mathcal{P}}} \cdot d_{\mathcal{P}} \cdot d(\mathfrak{P}|\mathcal{P}).$$

We therefore conclude that

$$\begin{aligned} |\mathfrak{B}_L| &= \sum_{\mu \in \Gamma} (t^{\mu} - 1) \\ &= 1 - [L : K] + \frac{1}{2} \sum_{\mathcal{P} \in \mathbb{P}_K} \frac{[L : K]}{e_{\mathcal{P}}} \cdot d_{\mathcal{P}} \cdot d(\mathfrak{P}|\mathcal{P}) \\ &= g_L. \end{aligned}$$

It follows that the set \mathfrak{B}_L of k -linearly independent homomorphic differentials forms a basis for the k -vector space Ω_L of holomorphic differentials of L . \square

Remark 4.2. 1. The quantities t^{μ} coincide with all of the Boseck invariants previously defined (see for example [3, 13, 23, 24, 29]). In particular, this agrees precisely with the invariants found in [13], which addresses cyclic automorphisms. This can be seen via the identity

$$\left\lfloor \frac{p^t u + v}{p^t n} \right\rfloor = \left\lfloor \frac{u}{n} + \frac{\left\lfloor \frac{v}{p^t} \right\rfloor}{n} \right\rfloor.$$

2. Theorem 4.1 remains true for towers such that the following quantities are independent of the choice of place \mathfrak{P} of L above a particular place \mathcal{P} of K , for all such places \mathcal{P} which ramify in L : the index of ramification $e(\mathfrak{P}|\mathcal{P})$, differential exponent $d(\mathfrak{P}|\mathcal{P})$, and inertia degree $f(\mathfrak{P}|\mathcal{P})$.
3. To satisfy the assumption that the place at infinity is unramified in L/K , it is sufficient to find any degree one place of K which is unramified in L .

We remind the reader that in the results of this section, it is not necessary to construct this basis in terms of an action of a generator of a cyclic group [29], nor is it necessary to assume that the field of constants k is algebraically closed. The basis \mathfrak{B}_L is defined completely in terms of the ramification data and those valuations arising from global standard form. In the next section, we will see that the basis of Theorem 4.1 is a useful construction for determining the Galois module structure of Ω_L .

5 Standard form

The difficulties presented by global standard form in towers do not particularly depend on whether a certain step in a tower is Kummer or Artin-Schreier; notably, Kummer extensions are no easier to handle than Artin-Schreier extensions in this respect. This includes even two-step towers, where each step consists of either kind of extension. In this section, we give an illustration of several cases where a global standard form may be obtained for towers. We note that these cases include all the bases which we have referenced in existing literature, as well as others. Furthermore, we elaborate on the various difficulties associated with finding a generator in global standard form. Currently, the only positive evidence that we have for

the non-existence of global standard form of generators in a tower derives from the fact that such a standard form does not exist for unramified extensions. It would be very useful if other examples of non-existence could be found.

5.1 Global standard form

We now turn our attention to the problem of when one may find a tower as given in §2. There are a few questions herein: first, when one may find the element x so that the place at infinity is unramified (which Boseck assumed for his preliminary constructions [3]), and also, when it is possible, given a function field L , to find the tower L/K with global standard form generators. Here, we give some examples of composita of function fields which satisfy the necessary criteria.

Example 5.1 (*Abelian extensions with easy conversion from composites into standard form towers*). Let L/K be an abelian extension of a rational field $K = k(x)$, with k perfect field of characteristic p . For the sake of this example, we suppose that the place at infinity in $k[x]$ is not ramified in L/K . This is important for the counting argument that yields the explicit basis. If l denotes the field of constants of L , then $L/l(x)$ is both abelian and geometric. Since generators in global standard form exist over the rational field [9], we may thus assume without loss of generality that L/K is geometric, that is, $k = l$. It is known that L/K is then a compositum of cyclic extensions of $k(x)$ (see for example [14]). Provided that k contains sufficient roots of unity, these cyclic extensions are either Kummer or generalised Artin-Schreier. We suppose that k contains the n_i th for of unity for each positive integer n_i ($(n_i, p) = 1$), for $i \in \{1, \dots, r\}$, where

$$\text{Gal}(L/K) = \mathbb{Z}/p^{t_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{t_s}\mathbb{Z} \times \mathbb{Z}/n_1\mathbb{Z} \times \dots \times \mathbb{Z}/n_r\mathbb{Z}.$$

Thus L/K may be written as a compositum of generalised Artin-Schreier extensions A_i/K with $\text{Gal}(A_i/K) \simeq \mathbb{Z}/p^{t_i}\mathbb{Z}$ for any $i \in \{1, \dots, s\}$ and Kummer extensions K_i/K with $\text{Gal}(K_i/K) \simeq \mathbb{Z}/n_i\mathbb{Z}$ for any $i \in \{1, \dots, r\}$. Via Hasse [9], any of the Kummer extensions K_i/K admits a generator y_i so that $y_i^{n_i} = c_i$ and y_i is in global standard form. As explained in the proof of Lemma 3.6, we cannot have partial ramification and equal ramification indices. Secondly, a generalised Artin-Schreier extension is a cyclic extension of degree a power of the characteristic of k , and it was proven by Madden [18] that such an extension may be expressed as a tower of Artin-Schreier extensions $A_i = A_{i,m_i}/A_{i,m_i-1}/\dots/A_{i,1} = K$, with for each $j = 1, \dots, m_i$ some generator $y_{i,j}$ of $A_{i,j}/A_{i,j-1}$, with defining equation $y_{i,j}^p - y_{i,j} = c_{i,j}$ in global standard form.

The following structure of L/K is natural for immediately arriving at global standard form in a tower from the composites of cyclic extensions over K . We suppose for any $i \neq j$, $i, j \in \{1, \dots, s\}$ that the ramified places of K in A_i are distinct from those in A_j . For any ramified place \mathcal{P} of K in L , we denote by \mathfrak{P} a place of L above \mathcal{P} , $e_{i,\mathcal{P}}$ the index of ramification of \mathcal{P} in the compositum $A_1 \dots A_s K_1 \dots K_r$. For such a place \mathcal{P} , we suppose that

$$(5) \quad n_i \nmid v_{\mathcal{P}}(c_i)e_{i-1,\mathcal{P}} \quad (i = 1, \dots, r),$$

where $v_{\mathcal{P}}(c_i)$ is the valuation of c_i in K . We also suppose that the quantities $v_{\mathcal{P}}(c_i)e_{i-1,\mathcal{P}}$ do not share a prime factor with n_i at every such ramified place \mathcal{P} , which as noted in §3.2 is needed for the existence of a global standard form. We denote

$$\tilde{A}_{i,j} = A_1 \dots A_{i-1} A_{i,1} \dots A_{i,j} \text{ and } \tilde{K}_i = A_1 \dots A_s K_1 \dots K_r.$$

Since the fields A_i ($i = 1, \dots, s$) do not share any ramified places, the Artin-Schreier generators $y_{i,j}$ of $\tilde{A}_{i,j}/\tilde{A}_{i,j-1}$ are automatically in global standard form. Similarly, by (5), y_i is a Kummer generator in global standard form of $\tilde{K}_i/\tilde{K}_{i-1}$. Therefore, all the conditions we required to create a k -basis as in Theorem 4.1 for Ω_L are now satisfied for the extension L/K .

Under the previous conditions, identification of the appropriate tower in global standard form for a compositum of generalised Artin-Schreier and Kummer extensions of the rational field is natural. Due to questions related to class numbers (see for example [22]), Hasse's method of obtaining global standard form is unclear to us in general, as it relies heavily upon the use of the principal ideal domain property of the field of rational functions. Nonetheless, we now give an example of a compositum of two Artin-Schreier extensions, which in contrast to Example 5.1 share ramified places, where global standard form is possible. Furthermore, in the following example, places may be either fully or partially ramified. We expect that other examples may also be produced.

Example 5.2 (*Elementary abelian extensions*). For this example, we suppose that L/K is now the compositum of two Artin-Schreier extensions L_1/K and L_2/K . We also suppose that the generators y_1 and y_2 of L_1 and L_2 , respectively, are in global standard form with

$$y_1^p - y_1 = a_1 + m_1 z^n,$$

and that

$$y_2^p - y_2 = m_2 z,$$

where $a_1, z \in K$, $m_1, m_2 \in k^*$, and $(n, p) = 1$. As k is perfect, there exists an element $\alpha \in k^*$ so that $m_1 m_2^{-n} = \alpha^p$. We suppose that a_1 and z do not share any places with negative valuation, i.e., if for some place \mathcal{P} of K , $v_{\mathcal{P}}(a_1) < 0$, then also $v_{\mathcal{P}}(z) \geq 0$, and vice versa. Therefore, at any ramified place \mathcal{P} of K in L , $v_{\mathcal{P}}(a_1 + m_1 z)$ is either equal to $v_{\mathcal{P}}(a_1)$ or $v_{\mathcal{P}}(z)$, and the ramified places of K in L_1 such that $v_{\mathcal{P}}(a_1 + m_1 z) = v_{\mathcal{P}}(a_1)$ are unramified in L_2 . For example, the simple equations

$$y_1^p - y_1 = \frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x} \text{ and } y_2^p - y_2 = \frac{1}{x}$$

are of this form and motivate these examples, as well as some comments on elementary abelian extensions, which we give below.

We let $\tilde{y}_1 = y_1 - \alpha y_2^n$. As defined, \tilde{y}_1 is an Artin-Schreier generator of L/L_2 in global standard form. We have

$$\begin{aligned} \tilde{y}_1^p - \tilde{y}_1 &= y_1^p - y_1 - m_1 m_2^{-n} (y_2 + m_2 z)^n + \alpha y_2^n \\ &= y_1^p - y_1 - m_1 m_2^{-n} \sum_{k=0}^n \binom{n}{k} (y_2)^k (m_2 z)^{n-k} + \alpha y_2^n \\ &= a_1 - m_1 m_2^{-n} \sum_{k=1}^n \binom{n}{k} (y_2)^k (m_2 z)^{n-k} + \alpha y_2^n \\ &= a_1 - m_1 m_2^{-n} \sum_{k=1}^{n-1} \binom{n}{k} (y_2)^k (m_2 z)^{n-k} + (\alpha - m_1 m_2^{-n}) y_2^n. \end{aligned}$$

Note that from Wu and Scheidler [32], we know that L/L_2 is ramified above any place \mathcal{P} of K such that $v_{\mathcal{P}}(a_1) < 0$ and that $v_{\mathcal{P}}(z) < 0$, unless $m_1 = m_2$ and $n = 1$. Clearly, at place \mathfrak{p}_2 of L_2 unramified in L , we have $v_{\mathfrak{p}_2}(\tilde{y}_1) \geq 0$. At the ramified places \mathfrak{p}_2 of L_2 in L above a place \mathcal{P} of K such that $v_{\mathcal{P}}(a_1) < 0$, we have $v_{\mathfrak{p}_2}(\tilde{y}_1) = v_{\mathcal{P}}(a_i)$. Thus \tilde{y}_1 is in standard form at \mathfrak{p}_2 as y_1 was at \mathcal{P} . At a ramified place \mathfrak{p}_2 of L_2 in L above a place \mathcal{P} of K such that $v_{\mathcal{P}}(z) < 0$, one has by the strict triangle inequality when $\alpha \neq \alpha^p$ that

$$\begin{aligned} v_{\mathfrak{p}_2}(\tilde{y}_1) &= \min\{v_{\mathfrak{p}_2}(a_1), v_{\mathfrak{p}_2}((y_2)^k(m_2z)^{n-k}), v_{\mathfrak{p}_2}(y_2^n)\} \\ &= \min\{v_{\mathfrak{p}_2}(a_1), v_{\mathcal{P}}(z)(k + p(n-k)) \ (k \in \{1, \dots, n\}), nv_{\mathcal{P}}(z)\} \\ &= v_{\mathcal{P}}(z)(1 + p(n-1)) < 0 \end{aligned}$$

and coprime to p . Note we find the same valuation as in [32, Corollary 3.10].

We conclude this section with an example of a p -elementary abelian extension of degree p^n , i.e., a Galois extension with Galois group equal to a product of n copies of $\mathbb{Z}/p\mathbb{Z}$, which is equivalent to an expression of an extension as compositum of n Artin-Schreier extensions. Let then $L/K = k(x)$ be an elementary abelian extension of degree p^n . It has been proven by Garcia and Stichtenoth [8] that as soon as k contains F_{p^n} , then there exists a generator y of L/K such that $y^{p^n} - y = z \in K$. In order to obtain a Boseck basis via this generator, we would need to have y expressed in a global standard form. By [8, Lemma 1.2], the elements y_{μ} ($\mu \in \mathbb{F}_{p^n}$) defined by $y_{\mu}^p - y_{\mu} = \mu z$ are precisely the Artin-Schreier generators of all of the subextensions of L of degree p over K . For any place \mathcal{P} of K ramified in L and $\mu \neq \mu'$, we have for the respective places \mathfrak{P}_{μ} and $\mathfrak{P}_{\mu'}$ of $K(y_{\mu})$ and $K(y_{\mu'})$ above \mathcal{P} that

$$(6) \quad v_{\mathfrak{P}_{\mu}}(y_{\mu}) = v_{\mathfrak{P}_{\mu'}}(y_{\mu'}) = v_{\mathcal{P}}(z).$$

If z is in standard form, then any places \mathcal{P} of K are either unramified or fully ramified [32, Theorem 3.11]. In particular, it is impossible to put a partially ramified place in standard form, even *locally*. Also, if we take the compositum of two Artin Schreier extensions of the form $y_1^p - y_1 = z_1$ and $y_2^p - y_2 = z_2$ with $z_1, z_2 \in K$ in global standard form and $v_{\mathcal{P}}(z_1) < v_{\mathcal{P}}(z_2)$ for some place \mathcal{P} of K ramified in L , then from the previous observation, \mathcal{P} is fully ramified [32]. For a basis $\{\mu_1, \mu_2\}$ of $\mathbb{F}_{p^2}/\mathbb{F}_p$, we may find a generator y of $K(y_1, y_2)/K$ with generating equation $y^{p^2} - y = z \in K$ by taking $y = \mu_1 y_1 + \mu_2 y_2$. However, the generator y cannot be written in global standard form at \mathcal{P} due to (6).

It is unknown to us when in this case a generator may be expressed globally in standard form in a natural way. To demonstrate, if a global standard form were possible in this case, then by [8], we would need to be able to obtain generators y_i of the Artin-Schreier extensions satisfying equations $y_i^p - y_i = z_i$ in global standard form, for all p -subextensions of L/K , so that for any $i \neq j$, there exists a $\mu_{i,j} \in \mathbb{F}_{p^n}^*$ so that $z_i = \mu_{i,j} z_j$. Particularly, the use of the generator y satisfying $y^{p^n} - y = z$ seems to be restrictive for obtaining a basis or Riemann-Hurwitz formula in terms of the valuation of the generator.

5.2 Weak standard form

In this section, we prove the existence of a weak standard form. This is an important ingredient in the proof of Theorem 6.1, which describes the $k[G]$ -module structure of Ω_L .

Lemma 5.1. *Let K be any function field of characteristic $p > 0$ with perfect field of constants k , and let L/K be an Artin-Schreier extension. Let $\{\mathfrak{p}_a\}_{a \in A}$ denote a finite set of places of K unramified in L .*

- (i) *There exists $\tilde{y}_0 \in L$ so that $L = K(\tilde{y}_0)$, the valuation of \tilde{y}_0 at each ramified place of L/K is negative and coprime to p , and the valuation of \tilde{y}_0 at each place of L above $\{\mathfrak{p}_a\}_{a \in A}$ is nonnegative.*
- (ii) *If an Artin-Schreier generator of L/K is in standard form at each ramified place, then the valuation of the generator at those places is uniquely determined. Precisely, with $z^p - z = u$ in standard form for each ramified place \mathcal{P} of K in L , one has for each such \mathcal{P} that*

$$v_{\mathfrak{P}}(z) = \max\{v_{\mathcal{P}}(u - (w^p - w)) \mid w \in K\},$$

where \mathfrak{P} denotes the place of L above \mathcal{P} .

Proof. We first prove the following: Let $\{\mathfrak{p}_t\}_{t \in T}$ be a finite set of places of L and $\{\mathfrak{p}_s\}_{s \in S} \subset \{\mathfrak{p}_t\}_{t \in T}$. If $u, v \in K \setminus \{0\}$, and $v_{\mathfrak{p}_s}(u) = v_{\mathfrak{p}_s}(v)$ for all $s \in S$, then there exists some $w \in L$ so that

- (1) $v_{\mathfrak{p}_t}(w) = 0$ for all $t \in T$; and
- (2) $v_{\mathfrak{p}_s}(u - w^p v) > v_{\mathfrak{p}_s}(u)$, for all $s \in S$.

For each place \mathfrak{p}_t , we let $\mathcal{O}_{\mathfrak{p}_t}$ denote the valuation ring at \mathfrak{p}_t and \mathfrak{p}_t the corresponding maximal ideal. By assumption, the residue class $\overline{(u/v)} \neq 0$ in $\mathcal{O}_{\mathfrak{p}_s}/\mathfrak{p}_s$, for each $s \in S$. As k is perfect, so too is $\mathcal{O}_{\mathfrak{p}_s}/\mathfrak{p}_s$ perfect, as a finite extension of k . This implies that for each $s \in S$, there exists some $w_s \in \mathcal{O}_{\mathfrak{p}_s}^*$ so that

$$\overline{w_s^p} = \overline{(u/v)} \in \mathcal{O}_{\mathfrak{p}_s}/\mathfrak{p}_s.$$

Let \mathfrak{p}^* be a place of K distinct from $\{\mathfrak{p}_t\}_{t \in T}$, which exists by the infinitude of places of K . By [30, Theorem 5.7.10], there exists an element $\alpha \in K$ so that

- (i) $v_{\mathfrak{p}^*}(\alpha) < 0$,
- (ii) and $v_{\mathfrak{p}}(\alpha) \geq 0$, for all $\mathfrak{p} \neq \mathfrak{p}^*$.

Let \mathcal{O} denote the integral closure of $k[\alpha]$ in K . By [30, Theorem 5.7.9],

$$\mathcal{O} = \bigcap_{\mathfrak{p} \neq \mathfrak{p}^*} \mathcal{O}_{\mathfrak{p}}.$$

As \mathcal{O} is a holomorphy ring [25, Proposition 3.2.9], for each place \mathfrak{p} of K with $\mathfrak{p} \neq \mathfrak{p}^*$,

$$\mathcal{O}/(\mathfrak{p} \cap \mathcal{O}) \simeq \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}.$$

In particular, this holds for $\mathfrak{p} = \mathfrak{p}_t$, for any $t \in T$. We now select for any $t \in T \setminus S$ a unit $a_t \in \mathcal{O}_{\mathfrak{p}_t}^*$. By the previous arguments and the Chinese Remainder Theorem [20, Theorem 1.3.6], we have

$$(7) \quad \mathcal{O}/\prod_{t \in T} (\mathfrak{p}_t \cap \mathcal{O}) \simeq \bigoplus_{t \in T} \mathcal{O}/(\mathfrak{p}_t \cap \mathcal{O}) \simeq \bigoplus_{t \in T} \mathcal{O}_{\mathfrak{p}_t}/\mathfrak{p}_t.$$

Via the isomorphism (7), we choose $w \in \mathcal{O}$ so that

- 1. $\overline{w} = \overline{w_s}$ for all $s \in S$, and

2. $\overline{w} = \overline{a}_t$ for all $t \in T \setminus S$.

First, we observe that this implies that the element w is a unit in $\mathcal{O}_{\mathfrak{p}_t}$, for all $t \in T$, so that the condition (1) is automatically satisfied. Second, we have for all $s \in S$ that

$$u/v = w_s^p = w^p \pmod{\mathfrak{p}_s}$$

from which it follows for all $s \in S$ that $v_{\mathfrak{p}_s}(u/v - w^p) > 0$. Thus condition (2) is also satisfied.

As the extension L/K is Artin-Schreier, it has a generator y such that $y^p - y = r \in K$. We let $\{\mathfrak{p}_s\}_{s \in S}$ denote the union of the set of places of K which ramify in L so that $p \mid v_{\mathfrak{p}_s}(r)$ with the set of all places of $\{\mathfrak{p}_a\}_{a \in A}$ which satisfy $v_{\mathfrak{p}_a}(r) < 0$. Also, we let $\{\mathfrak{p}_t\}_{t \in T}$ denote the union of the set of all places of K which ramify in L/K with $\{\mathfrak{p}_a\}_{a \in A}$. For any of the unramified places \mathfrak{p}_a and any place \mathfrak{P} of L above \mathfrak{p}_a , we have if $v_{\mathfrak{p}_a}(r) < 0$,

$$pv_{\mathfrak{P}}(y) = v_{\mathfrak{P}}(y^p - y) = v_{\mathfrak{P}}(r) = v_{\mathfrak{p}_a}(r),$$

so that $p \mid v_{\mathfrak{p}_a}(r)$ whenever $v_{\mathfrak{p}_a}(r) < 0$. Via weak approximation [30, Theorem 2.5.3], we may find an element $\beta \in K$ so that $v_{\mathfrak{p}_s}(\beta) = v_{\mathfrak{p}_s}(r)/p$, for all $s \in S$, and so that $v_{\mathfrak{p}_s}(\beta) = 0$, for all $t \in T \setminus S$. In particular, we have for all $s \in S$ that $v_{\mathfrak{p}_s}(\beta^p) = v_{\mathfrak{p}_s}(r)$, for all $s \in S$. By the first part of this proof, we find an element $w \in K$ so that $v_{\mathfrak{p}_t}(w) = 0$ for all $t \in T$, and so that

$$v_{\mathfrak{p}_s}(r - w^p \beta^p) > v_{\mathfrak{p}_s}(\beta^p),$$

for all $s \in S$. Thus $v_{\mathfrak{p}_s}(\beta w) = v_{\mathfrak{p}_s}(r)/p$ for all $s \in S$ and, for all $t \in T \setminus S$, we have $v_{\mathfrak{p}_t}(\beta w) = 0$. In particular, we find for all $s \in S$ that

$$v_{\mathfrak{p}_s}(r - ((\beta w)^p - (\beta w))) \geq \min\{v_{\mathfrak{p}_s}(r - (\beta w)^p), v_{\mathfrak{p}_s}(\beta w)\} > v_{\mathfrak{p}_s}(r),$$

and for all $t \in T \setminus S$, we find that $v_{\mathfrak{p}_t}(r - ((\beta w)^p - (\beta w))) = v_{\mathfrak{p}_t}(r)$, as $v_{\mathfrak{p}_t}(r) < 0$. Choosing the element $y' = y + tw$ yields a new Artin-Schreier generation of L/K , where the valuations are strictly greater than those for the original y at all places of $\{\mathfrak{p}_s\}_{s \in S}$, i.e., those places of $\{\mathfrak{p}_t\}_{t \in T}$ for which the valuation of y is negative and divisible by p . Applying this replacement algorithm successively eventually terminates, which yields an Artin-Schreier generator $\tilde{y}_0^p - \tilde{y}_0 = \tilde{r}_0$ for which $v_{\mathfrak{p}}(\tilde{y}_0) < 0$ and coprime with p for each place \mathfrak{p} of K ramified in L , and for which $v_{\mathfrak{p}_a}(\tilde{y}_0) \geq 0$ for all $a \in A$.

Part (ii) follows as in the proof of Lemma 3.7.7 and Proposition 3.7.8 of [25]. \square

As a corollary, we find that the valuations at the unramified places we choose may be made equal to zero in weak standard form. This is not necessary for the proof of Theorem 6.1. This form was mentioned by Valentini and Madan in [29], who gave it when the field of constants is algebraically closed; we include the result out of independent interest in when this form exists if the field of constants is only assumed to be perfect.

Corollary 5.2. *Let K be any function field of characteristic $p > 0$ with perfect field of constants $k \neq \mathbb{F}_p$, and let L/K be an Artin-Schreier extension. Let $\{\mathfrak{p}_a\}_{a \in A}$ denote a finite set of places of K unramified in L . There exists $\tilde{y} \in L$ so that $L = K(\tilde{y})$, the valuation of \tilde{y} at each ramified place of L/K is negative and coprime to p , and the valuation of \tilde{y} at each place of L above $\{\mathfrak{p}_a\}_{a \in A}$ is equal to zero.*

Proof. We let $\{\mathfrak{p}_t\}_{t \in T}$ denote the union of the places of K which ramify in L with the set of unramified places $\{\mathfrak{p}_a\}_{a \in A}$. We assume the notation of the proof of Lemma 5.1, and with that notation, we let $S = A$. Furthermore, we also let \tilde{r}_0 and \tilde{y}_0 be as in Lemma 5.1.

The set S may be partitioned into three subsets $S = S_1 \cup S_2 \cup S_3$, which are defined in the following way. The set S_1 denotes those places of $\{\mathfrak{p}_s\}_{s \in S}$ for which $v_{\mathfrak{p}_s}(\tilde{r}_0) > 0$. The set S_2 denotes those places of $\{\mathfrak{p}_s\}_{s \in S}$ for which $v_{\mathfrak{p}_s}(\tilde{r}_0) = 0$ and \tilde{r}_0 lies in the image of the Artin-Schreier map $x \rightarrow x^p - x$ in the residue field of K at \mathfrak{p}_s . The set S_3 denotes the places of S not belonging to S_1 or S_2 . For each $s \in S_2$, let $x_s \in \mathcal{O}_{\mathfrak{p}_s}$ be an element so that

$$\tilde{r}_0 \equiv \bar{x}_s^p - \bar{x}_s \pmod{\mathfrak{p}_s},$$

which exists for each such s by definition of S_2 . We note that the element x_s is necessarily a unit. By the Chinese remainder theorem [20, Theorem 1.3.6], let α be an element of \mathcal{O} (with \mathcal{O} defined as before) so that

1. $\bar{\alpha} \equiv 0 \pmod{\mathfrak{p}_s}$, for all $s \in S_1$;
2. $\bar{\alpha} \equiv \bar{x}_s \pmod{\mathfrak{p}_s}$, for all $s \in S_2$; and
3. $\bar{\alpha} \not\equiv 0 \pmod{\mathfrak{p}_s}$, for all $s \in S_3$.

As $\alpha \in \mathcal{O}$, the element $\tilde{r}_0 - (\alpha^p - \alpha)$ retains its negative and coprime to p valuation at the places of K which ramify in L . For all $s \in S_1$,

$$v_{\mathfrak{p}_s}(\tilde{r}_0 - (\alpha^p - \alpha)) \geq \min\{v_{\mathfrak{p}_s}(\tilde{r}_0), v_{\mathfrak{p}_s}(\alpha^p), v_{\mathfrak{p}_s}(\alpha)\} > 0.$$

For all $s \in S_2$,

$$\tilde{r}_0 - (\alpha^p - \alpha) \equiv \tilde{r}_0 - (x_s^p - x_s) \equiv 0 \pmod{\mathfrak{p}_s},$$

as $\bar{\alpha}^p - \bar{\alpha} \equiv \bar{x}_s^p - \bar{x}_s \pmod{\mathfrak{p}_s}$, so that $v_{\mathfrak{p}_s}(\tilde{r}_0 - (\alpha^p - \alpha)) > 0$. For all $s \in S_3$,

$$v_{\mathfrak{p}_s}(\tilde{r}_0 - (\alpha^p - \alpha)) = 0,$$

as \tilde{r}_0 does not lie in the image of the Artin-Schreier map of the residue field at \mathfrak{p}_s . Choosing $\gamma \in k \setminus \mathbb{F}_p$ (as $k \neq \mathbb{F}_p$) now yields that the element $\tilde{r} := \tilde{r}_0 - (\alpha^p - \alpha) - (\gamma^p - \gamma)$ has negative and coprime to p valuation at the places of K which ramify in L . As $\gamma^p - \gamma \neq 0$ in the residue field at \mathfrak{p} for any place \mathfrak{p} of K , it follows for all $s \in S_1$ that

$$v_{\mathfrak{p}_s}(\tilde{r}) = \min\{v_{\mathfrak{p}_s}(\tilde{r}_0 - (\alpha^p - \alpha)), v_{\mathfrak{p}_s}(\gamma^p - \gamma)\} = 0.$$

By the same reasoning, the same holds for all $s \in S_2$. Finally, for all $s \in S_3$, as \tilde{r}_0 does not lie in the image of the Artin-Schreier map in the residue field, it follows that

$$\tilde{r}_0 \not\equiv (\alpha + \gamma)^p - (\alpha + \gamma) = (\alpha^p - \alpha) + (\gamma^p - \gamma) \pmod{\mathfrak{p}_s},$$

and hence for such s that

$$\begin{aligned} v_{\mathfrak{p}_s}(\tilde{r}) &= v_{\mathfrak{p}_s}(\tilde{r}_0 - (\alpha^p - \alpha) - (\gamma^p - \gamma)) \\ &= v_{\mathfrak{p}_s}(\tilde{r}_0 - ((\alpha + \gamma)^p - (\alpha + \gamma))) \\ &= 0. \end{aligned}$$

Therefore, the element \tilde{r} , and thus the associated Artin-Schreier generator

$$\tilde{y} = \tilde{y}_0 + (\alpha + \gamma),$$

are as desired. \square

For completeness, we state and give a proof of the analogue for weak standard form in Kummer extensions.

Lemma 5.3. *Let L/K be a Kummer extension of function fields of degree n , with constant field of positive characteristic $p > 0$ which contains all n th roots of unity. Let $\{\mathfrak{p}_a\}_{a \in A}$ denote a finite set of places of K which are unramified in L . There exists $\tilde{y} \in L$ so that $L = K(\tilde{y})$, the valuation of \tilde{y} at all places above those of $\{\mathfrak{p}_a\}_{a \in A}$ is equal to zero, and the valuation of \tilde{y} at all places of K which ramify in L lies in the set $\{1, \dots, n-1\}$.*

Proof. As k contains the n th roots of unity, there exists $y \in L$ so that $L = K(y)$ and $y^n = c \in K$. At each ramified place \mathfrak{p} of K , we have

$$v_{\mathfrak{p}}(c) = l_{\mathfrak{p}} + nq_{\mathfrak{p}}$$

for some integer $q_{\mathfrak{p}}$ and $l_{\mathfrak{p}} \in \{1, \dots, n-1\}$, and for each $a \in A$, we have $v_{\mathfrak{p}_a}(c) = nq_{\mathfrak{p}_a}$ for some integer $q_{\mathfrak{p}_a}$ [25, Proposition 3.7.3]. By weak approximation, we choose an element $\alpha \in K$ so that

1. $v_{\mathfrak{p}}(\alpha) = -q_{\mathfrak{p}}$, for all places of K which ramify in L ; and
2. $v_{\mathfrak{p}_a}(\alpha) = -q_{\mathfrak{p}_a}$, for all $a \in A$.

It follows that the element $\tilde{y} = \alpha y$, which satisfies $\tilde{y}^n = (\alpha y)^n = \alpha^n c$, is as desired. \square

We now prove a Lemma for Artin-Schreier extensions, which establishes a special form of the strict triangle inequality.

Lemma 5.4. *Let L/K be a cyclic, geometric extension of function fields of degree p , with constant field k of characteristic $p > 0$. Let \mathfrak{p} be a place of K . Suppose that the Artin-Schreier generator $y^p - y = r \in K$ of L/K is in local standard form at \mathfrak{p} , i.e., that for all places \mathfrak{P} of L above \mathfrak{p} , $v_{\mathfrak{P}}(y) < 0$ and coprime to p if \mathfrak{p} is ramified in L , and $v_{\mathfrak{P}}(y) \geq 0$ if \mathfrak{p} is unramified in L . For $a \in L$, let*

$$a = b_0 + b_1 y + \dots + b_{p-1} y^{p-1}, \quad b_0, b_1, \dots, b_{p-1} \in K.$$

There exists a place \mathfrak{P} of L above \mathfrak{p} so that $v_{\mathfrak{P}}(a) = m$, where

$$m = \begin{cases} \min_{0 \leq j < p} \{v_{\mathfrak{P}}(b_j y^j)\} & \text{if } \mathfrak{p} \text{ is ramified in } L \\ \min_{0 \leq j < p} \{v_{\mathfrak{p}}(b_j)\} & \text{if } \mathfrak{p} \text{ is unramified in } L \end{cases}$$

Proof. The proof is precisely as in [17, Lemma 2]. \square

Our corollary to the Lemma 5.4 is the natural extension to generalised Artin-Schreier extensions. One may compare this to [29, Lemma 2]; we remark that this corollary differs slightly from the aforementioned result, as the generators of unramified steps in the tower do not appear, so that we only need to require the weak standard form given by Lemma 5.1. This is done so that we may work with a version of Theorem 1 of [29] over a general perfect constant field.

Lemma 5.5. *Let L/K be a cyclic, geometric extension of function fields of degree p^n , with constant field k of characteristic $p > 0$ and Artin-Schreier tower $L = L_n/L_{n-1}/\dots/L_0 = K$. Let \mathfrak{p} be a place of K , and let \mathfrak{p}_{i-1} be a place of L_{i-1} above \mathfrak{p} . Suppose for each $i = 1, \dots, n$ and each such place \mathfrak{p}_{i-1} that the Artin-Schreier generator $y_i^p - y_i = r_i \in L_{i-1}$ of L_i/L_{i-1} is in standard*

form at \mathfrak{p}_{i-1} if \mathfrak{p}_{i-1} is ramified in L_i , or that $v_{\mathfrak{P}}(y) \geq 0$ for all \mathfrak{P} of L_i above \mathfrak{p}_{i-1} if \mathfrak{p}_{i-1} is unramified in L_i . For $a \in L$, let

$$a = \sum_{\mu_1, \dots, \mu_n} a_{\mu_1, \dots, \mu_n} y_1^{\mu_1} \cdots y_n^{\mu_n}, \quad a_{\mu_1, \dots, \mu_n} \in K.$$

Then

$$\min_{\mathfrak{P}|\mathfrak{p}} v_{\mathfrak{P}}(a) = \min_{\substack{\mathfrak{P}|\mathfrak{p} \\ \mu_i \\ \mathfrak{p}_i|\mathfrak{p}_{i-1} \text{ ramifies}}} v_{\mathfrak{P}} \left(a_{\mu_1, \dots, \mu_n} \prod_i y_i^{\mu_i} \right),$$

where the minimum is taken over all places \mathfrak{P} of L over \mathfrak{p} and $\mu_i \in \{0, \dots, p-1\}$, for all i where $\mathfrak{p}_i|\mathfrak{p}_{i-1}$ ramifies.

Proof. This follows by induction from Lemma 5.4. \square

6 Galois action on Ω_L for abelian extensions

6.1 Galois module structure of Ω_L for cyclic extensions

As Lemma 5.3, Lemma 5.5, and the results of Tamagawa [26] are both valid over a perfect field, and the standard form of Lemma 5.1 is enough to prove Theorem 1 of [29], we find from the same argument as [13] that Theorem 7 of [13] holds over any perfect field of characteristic $p > 0$. More precisely, let K be any function field and k a perfect field of characteristic $p > 0$. Let L/K be a cyclic Galois extension of degree $p^t n$ with $(n, p) = 1$. Denote by $G = G_p \times G_n$ the cyclic Galois group with generator σ of L/K and its unique cyclic p -Sylow G_p with generator $\sigma_p = \sigma^n$ and $G_n \simeq \mathbb{Z}/n\mathbb{Z} \simeq G/G_p$, with generator $\sigma_n = \bar{\sigma}$, where $\bar{\sigma}$ denotes the image of σ in G/G_p . Thus, L/L^{G_p} is a generalised Artin-Schreier extension, which we may write as a tower $L = A_t/A_{t-1}/\cdots/A_0 = L^{G_p}$, where A_i/A_{i-1} are Artin-Schreier extensions and L^{G_p}/K is a Kummer extension. Denote by \mathbb{P}_K the set of places of K which ramify in L . For a place \mathcal{P} of K , we denote by \mathfrak{P} of L above \mathcal{P} , $\mathfrak{p}_{A_i} = \mathfrak{P} \cap A_i$, and $\mathfrak{p}_{Ku} = \mathfrak{P} \cap L^{G_p}$. By Lemma 5.1, we may choose an Artin-Schreier generator y_{A_i} of A_i/A_{i-1} such that $y_{A_i}^p - y_{A_i} = c_{A_i}$, where for any place of A_{i-1} unramified in A_i above a ramified place of K in L , $v_{\mathcal{P},i} := v_{\mathfrak{p}_{A_{i-1}}}(c_{A_i}) \geq 0$, and such that for any place of A_{i-1} ramified in A_i , $v_{\mathcal{P},i} < 0$ and coprime with p . Up to a base change of the form $y_{A_i} \rightarrow l_{A_i} y_{A_i}$ with $l_{A_i} \in \mathbb{F}_p$, we may suppose for simplicity that $\sigma_p^{p^{i-1}}(y_{A_i}) = y_{A_i} + 1$. By Lemma 5.3, we may choose a Kummer generator $y_{Ku}^n = c_{Ku}$ such that for any place \mathcal{P} of K unramified in L^{G_p} but ramified in L , $v_{\mathcal{P}}(c_{Ku}) = 0$, and such that for any place of K ramified in L^{G_p} , $v_{\mathcal{P}}(c_{Ku}) > 0$. We denote by $v_{\mathcal{P},Ku} = v_{\mathfrak{p}_{Ku}}(y_{Ku})$. We recall that we may identify G_n with the group generated by a primitive n -root of unity ξ , so that up to choosing another generator of G , we may suppose that $\sigma_n(y_{Ku}) = \xi y_{Ku}$. As L/L^{G_p} is abelian, we may then decompose it as $L/L^{p,unr}/L^{G_p}$, where $L^{p,unr}/L^{G_p}$ is unramified of degree, say, $t_{unr} \leq t$, and such that for any $t_{unr} + 1 \leq i \leq t$, there is at least one ramified place in the extension A_i/A_{i-1} . (For this, take $L^{p,unr}$ to be the fixed field of the product of the inertia group at the ramified places). Let

$$\mu = (\mu_{A_t}, \dots, \mu_{A_1}, \mu_{Ku}) \in \prod_{i=1}^t \{0, \dots, p-1\} \times \{0, \dots, n-1\}$$

and $\mu_p = \mu_{A_1} + p\mu_{A_2} + \cdots + p^{t-1}\mu_{A_t}$. We can identify μ_p with $(\mu_{A_t}, \dots, \mu_{A_1})$, and we therefore write

$$\mu = (\mu_p, \mu_{Ku}) \in \Gamma := \{0, \dots, p^t - 1\} \times \{0, \dots, n - 1\}.$$

Let $\lambda_{\mathcal{P}}^{\mu}$ and $\rho_{\mathcal{P}}^{\mu}$ be defined as in Theorem 4.1, i.e.,

$$e_{\mathcal{P}}\lambda_{\mathcal{P}}^{\mu} + \rho_{\mathcal{P}}^{\mu} = \left[\sum_{i=t_{unr}}^t e(\mathfrak{P}|\mathfrak{p}_{A_i})((p-1-\mu_{A_i})(-v_{p,i}) + (p-1)) \right] + e(\mathfrak{P}|\mathfrak{p}_{Ku})(\mu_{Ku}v_{\mathcal{P},Ku} + (e(\mathfrak{p}_{Ku}|\mathcal{P}) - 1)),$$

with $0 \leq \rho_{\mathcal{P}}^{\mu} \leq e_{\mathcal{P}} - 1$. As before, we let

$$(8) \quad t^{\mu} = \sum_{\mathcal{P} \in \mathbb{P}_K} d_{\mathcal{P}} \left(\lambda_{\mathcal{P}}^{\mu} - \sum_{i \in R_0(\mathcal{P})} \frac{e(\mathfrak{P}|\mathfrak{p}_i)}{e_{\mathcal{P}}} v_{\mathcal{P},i} \mu_i \right).$$

This is the same invariant as in Definition 4 of [13] (see also Remark 4.2). We also note that when k is algebraically closed, $d_{\mathcal{P}} = 1$.

Finally, we define the $k[G]$ -indecomposable module Δ_{μ} to be the μ_p -dimensional K -vector space with basis $\{v_1, \dots, v_{\mu_p}\}$ and Galois action given by $\sigma(v_i) = \xi^{\mu_{Ku}} v_i + v_{i+1}$ for all $1 \leq i \leq \mu_p - 1$ and $\sigma(v_{\mu_p}) = \xi^{\mu_{Ku}} v_{\mu_p}$.

Theorem 6.1. *In the previous context, we obtain the following the decomposition in indecomposable Galois $k[G]$ -modules:*

$$\Omega_L \simeq \bigoplus_{\mu \in \Gamma} \Delta_{\mu}^{d_{\mu}},$$

where d_{μ} denotes the number of times that the decomposition Δ_{μ} appears in Ω_L . We let $t^{(\alpha, \beta)}$ be defined according to (8) for a tuple $(\alpha, \beta) \in \Gamma$. For each $\mu_{Ku} \in \{0, \dots, n - 1\}$, and

$$1. \ 0 \leq \mu_p < p^t - p^{t_{unr}},$$

$$d_{\mu} = t^{(\mu_p-1, \mu_{Ku})} - t^{(\mu_p, \mu_{Ku})} + \delta_{(\mu_p-1, \mu_{Ku})} - \delta_{(\mu_p, \mu_{Ku})}$$

$$2. \ \mu_p = p^t - p^{t_{unr}}, \\ - \text{ for } \mu_{Ku} \neq 0,$$

$$d_{\mu} = t^{(\mu_p-1, \mu_{Ku})} - \frac{1}{p^{t_{unr}}} t^{(\mu_p, \mu_{Ku})} + \delta_{(\mu_p-1, \mu_{Ku})}$$

$$- \text{ for } \mu_{Ku} = 0,$$

$$d_{\mu} = t^{(\mu_p-1, \mu_{Ku})} - \delta_{(\mu_p-1, \mu_{Ku})} - 1$$

$$3. \ p^t - p^{t_{unr}} < \mu_p \leq p^t,$$

$$(a) \ t_{unr} \neq 0, \\ - \text{ for } \mu_{Ku} = 0 \text{ and } \mu_p = p^t - p^{t_{unr}} + 1,$$

$$d_{\mu} = 1$$

– for $\mu_p = p^t$,

$$d_\mu = \frac{1}{p^{t_{unr}}} (g_{L^p, \mu_{unr}} - 1 + t^{(\mu_p, \mu_{Ku})})$$

– $d_\mu = 0$, otherwise.

(b) $t_{unr} = 0$ and $\mu_p = p^l$,

$$d_\mu = g_K - 1 + t^\mu + \delta_\mu$$

where we define the integers δ_μ so that $\delta_\mu = 1$ if $t^\mu = 0$ and $\delta_\mu = 0$ otherwise.

6.2 Galois action on Ω_L

We now suppose that L/K is Galois of degree n with Galois group G and perfect constant field k of characteristic $p > 0$. We examine the Galois module structure of Ω_L (i.e., as a $k[G]$ -module) in the case that L/K is abelian. We note that it could be possible to formulate similar results in certain non-abelian cases, if we are able to find a tower to satisfy the requisite invariants of Theorem 4.1 and we know the action of the Galois group on the generator of this tower. For the moment, we simply suppose that the extension L/K is abelian. Since G is abelian, we may write L/K as a tower $L/L^{G_p}/K$, where L/L^{G_p} is the maximal subextension of prime power degree p^t , and L^{G_p}/K is an extension of degree d coprime to p . We denote $\text{Gal}(L/L^{G_p}) = G_p$ and $\text{Gal}(L^{G_p}/K) = G_d$. Thus,

$$G := \text{Gal}(L/K) \simeq G_p \times G_d,$$

with

$$G_p := \mathbb{Z}/p^{t_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{t_s}\mathbb{Z} \text{ and } G_d := \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}.$$

Suppose that we have a tower $L/L^{G_p} = K_r/\cdots/K_0 = K$, with K_i/K_{i-1} a Kummer extension with Galois group isomorphic to $\mathbb{Z}/n_i\mathbb{Z}$, for each $i = 1, \dots, r$. Suppose also that we have Kummer generators for each Kummer extension K_i/K_{i-1} with $y_{K_i}^{n_i} = c_{K_i}$. Given an n_i th root of unity ξ_i , we know that $\text{Gal}(K_i/K_{i-1})$ may be identified with the group generated by ξ_i , and that the action of ξ_i on y_i is given by $y_{K_i} \rightarrow \xi_i y_{K_i}$. Moreover, by Galois theory, the p -extension L/L^{G_p} may be expressed as a tower

$$L = A_s/\cdots/A_0 = L^{G_p}$$

where A_i/A_{i-1} is a generalised Artin-Schreier extension ($i = 1, \dots, s$) with a unique decomposition

$$A_i = A_{i, \mu_i}/\cdots/A_{i, 0} = A_{i-1},$$

where $A_{i, j}/A_{i, j-1}$ is an Artin-Schreier extension with an Artin-Schreier generator $y_{A_{i, j}}$ such that $y_{A_{i, j}}^p - y_{A_{i, j}} = c_{A_{i, j}}$, for any $j \in \{0, \dots, \mu_i\}$. We denote by σ_{A_i} a generator of A_i/A_{i-1} . We may furthermore suppose that $\sigma_{A_i}^{p^{i-1}}(y_{A_{i, j}}) = y_{A_{i, j}} + 1$. We let $i \in \{1, \dots, s\}$ and $\mu_{A_i} \in \{0, \dots, p^{t_i} - 1\}$, with

$$\mu_{A_i} = \mu_{A_{i, 1}} + p\mu_{A_{i, 2}} + \cdots + p^{t_i-1}\mu_{A_{i, \mu_i}},$$

$i \in \{1, \dots, r\}$, and $\mu_{K_i} \in \{0, \dots, d_i - 1\}$. We also let $\mu = (\mu_{A_1}, \dots, \mu_{A_s}, \mu_{K_1}, \dots, \mu_{K_r})$,

$$z_\mu = y_{A_1}^{\mu_{A_1}} \cdots y_{A_s}^{\mu_{A_s}} y_{K_1}^{\mu_{K_1}} \cdots y_{K_r}^{\mu_{K_r}}, \text{ and } y_{A_i}^{\mu_{A_i}} = y_{A_{i, \mu_i}}^{\mu_{A_{i, \mu_i}}} \cdots y_{A_{i, 1}}^{\mu_{A_{i, 1}}}.$$

We recall that the action on the tower may be expressed more precisely as follows.

Lemma 6.2 (Proposition 1, [24]). *For $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, u_i\}$,*

$$\sigma_i(y_{A_{i,j}}) = y_{A_{i,j}} + f_{A_{i,j}}(y_{A_{i,1}}, \dots, y_{A_{i,j-1}}),$$

for some $f_{A_{i,j}}(T_1, \dots, T_{j-1}) \in \mathbb{Z}[T_1, \dots, T_{j-1}]$.

Lemma 6.3 (Proposition 4, [24]). *For any*

$$\mu = (\mu_{A_1}, \dots, \mu_{A_s}, \mu_{K_1}, \dots, \mu_{K_r}) \in \prod_{i=1}^s \{0, \dots, p^{t_i} - 1\} \times \prod_{i=1}^r \{0, \dots, n_i - 1\}$$

and $i \in \{1, \dots, s\}$,

$$(\sigma_i - 1)^{\mu_{A_i}}(z^\mu) = y_{A_1}^{\mu_{A_1}} \cdots y_{A_{i-1}}^{\mu_{A_{i-1}}} \mu_{A_i}! y_{A_{i+1}}^{\mu_{A_{i+1}}} \cdots y_{A_s}^{\mu_{A_s}} y_{K_1}^{\mu_{K_1}} \cdots y_{K_r}^{\mu_{K_r}}.$$

Furthermore,

$$(\sigma_i - 1)^{\mu_{A_i}+1}(z^\mu) = 0.$$

These all follow as in [24], to which we refer the reader for further details. We now additionally suppose that the extension L/K is geometric, so that L has constant field equal to k , all of the generators $y_{A_{i,j}}$ and y_{K_i} are expressed in global standard form, $K = k(x)$ is a rational field such that the place at infinity of $k[x]$ is unramified, and the usual condition on equality of valuations for Kummer generators, i.e., that the valuations of c_{K_i} are not equal and share a prime factor with n_i for all ramified places of K_i/K_{i-1} . With the notation of §4, we thus obtain the basis

$$\mathfrak{B}_L = \{w_{v,\mu} = x^\nu [g_\mu(x)]^{-1} z^\mu dx \mid \mu \in \Gamma, 0 \leq \nu \leq t^\mu - 2\}.$$

We first prove a lemma which describes explicitly the Galois action on the basis \mathfrak{B}_L .

Lemma 6.4. *For any $w_{v,\mu} \in \mathfrak{B}_L$ and*

$$h = (h_{A_1}, \dots, h_{A_s}, h_{K_1}, \dots, h_{K_r}) \in \prod_{i=1}^s \{0, \dots, p^{t_i} - 1\} \times \prod_{i=1}^r \{0, \dots, n_i - 1\},$$

we have

$$\sigma_{A_1}^{h_{A_1}} \cdots \sigma_{A_s}^{h_{A_s}} \sigma_{K_1}^{h_{K_1}} \cdots \sigma_{K_s}^{h_{K_s}}(w_{\mu,\nu}) = \sum_{\substack{\mu' \in \Gamma, \mu'_i \leq \mu_{A_i}, i \in \{1, \dots, s\} \\ (\mu_{K_1}, \dots, \mu_{K_r}) = (\mu'_{K_1}, \dots, \mu'_{K_r})}} c_{\mu,\mu',h} \sum_{l=0}^{t^{\mu'} - t^\mu} B_{\mu,\mu',l} w_{\mu',\nu+l},$$

where $c_{\mu,\mu',h}, B_{\mu,\mu',l} \in K$.

Proof. By definition of σ_{A_i} and σ_{K_i} , we have

$$\begin{aligned} & \sigma_{A_1}^{h_{A_1}} \cdots \sigma_{A_s}^{h_{A_s}} \sigma_{K_1}^{h_{K_1}} \cdots \sigma_{K_s}^{h_{K_s}}(w_{\mu,\nu}) \\ &= x^\nu [g_\mu(x)]^{-1} (\sigma_{A_1}^{h_{A_1}}(y_{A_{1,1}}))^{\mu_{A_1,1}} \cdots (\sigma_{A_1}^{p^{t_1}-1} h_{A_1}(y_{A_{1,t_1}}))^{\mu_{A_1,t_1}} \cdots (\sigma_{A_2}^{h_{A_2}}(y_{A_{s,1}}))^{\mu_{A_s,1}} \cdots (\sigma_{A_s}^{p^{t_s}-1} h_{A_s}(y_{A_{s,t_s}}))^{\mu_{A_s,t_s}} \\ & \quad \cdots (\sigma_{K_1}^{h_{K_1}}(y_{K_1}))^{\mu_{K_1}} \cdots (\sigma_{K_r}^{h_{K_r}}(y_{K_r}))^{\mu_{K_r}} dx \\ &= x^\nu [g_\mu(x)]^{-1} (y_{A_{1,1}} + h_{A_1})^{\mu_{A_1,1}} \cdots (y_{A_{1,t_1}} + P_{A_{1,t_1}}(y_{A_{1,1}}, \dots, y_{A_{1,t_1-1}}))^{\mu_{A_1,t_1}} \cdots \\ & \quad \cdots (y_{A_{s,1}} + h_{A_s})^{\mu_{A_s,1}} \cdots (y_{A_{s,n_s}} + P_{A_{s,n_s}}(y_{A_{s,1}}, \dots, y_{A_{s,n_s-1}}))^{\mu_{A_s,n_s}} (\xi_1^{h_{K_1}} y_1)^{\mu_{s+1}} \cdots (\xi_s^{h_{K_s}} y_r)^{\mu_{s+r}} dx \\ &= x^\nu [g_\mu(x)]^{-1} \sum_{\mu'} c_{\mu,\mu',h} y_{A_{1,1}}^{\mu'_{A_1,1}} \cdots y_{A_{s,t_s}}^{\mu'_{A_s,t_s}} y_{K_1}^{\mu'_{K_1}} \cdots y_{K_r}^{\mu'_{K_r}} dx, \end{aligned}$$

where $P_{A_i,j}(T_1, \dots, T_{j-1}) \in \mathbb{Z}[T_1, \dots, T_{j-1}]$. By the proof of Theorem 2 of [24], one can show that $\mu'_{A_i} \leq \mu_{A_i}$, and we thus have that $\lambda_{\mathcal{P}}^{\mu'} \geq \lambda_{\mathcal{P}}^{\mu}$. We then set

$$h_{\mu,\mu'}(x) = \frac{g_{\mu'}(x)}{g_{\mu}(x)} = \prod_{\mathcal{P} \in \mathbb{P}_K} p_{\mathcal{P}}(x)^{\lambda_{\mathcal{P}}^{\mu'} - \lambda_{\mathcal{P}}^{\mu}} = \sum_{l=0}^{t^{\mu'} - t^{\mu}} B_{\mu,\mu',l} x^l$$

Then

$$\sigma_{A_1}^{h_{A_1}} \cdots \sigma_{A_s}^{h_{A_s}} \sigma_{K_1}^{h_{K_1}} \cdots \sigma_{K_s}^{h_{K_s}}(w_{\mu,\nu}) = \sum_{\substack{\mu' \in \Gamma, \mu'_{A_i} \leq \mu_{A_i}, i \in \{1, \dots, s\} \\ (\mu_{K_1}, \dots, \mu_{K_r}) = (\mu'_{K_1}, \dots, \mu'_{K_r})}} c_{\mu,\mu',h} \sum_{l=0}^{t^{\mu'} - t^{\mu}} B_{\mu,\mu',l} w_{\mu',\nu+l}.$$

We also have that

$$\nu + l \leq t^{\mu} + t^{\mu'} - t^{\mu} - 2 \leq t^{\mu'} - 2.$$

As a consequence, $w_{\mu',\nu+l} \in \mathfrak{B}_L$. \square

Theorem 6.5. *With the notation of the previous lemma, let $\theta_{\mu'}^{\mu,\nu} := \sum_{l=0}^{t^{\mu'} - t^{\mu}} B_{\mu,\mu',l} w_{\mu',\nu+l}$. The set*

$$\{\theta_{\mu'}^{\mu,\nu} \mid \text{with } \mu'_{A_i} \leq \mu_{A_i} \text{ for } i \in \{1, \dots, s\} \text{ and } (\mu_{K_1}, \dots, \mu_{K_r}) = (\mu'_{K_1}, \dots, \mu'_{K_r})\}$$

generates a $k[G]$ -submodule $U_{\mu,\nu}$ isomorphic to $k[G]w_{\mu,\nu}$ of dimension $\prod_{i=1}^s (\mu_{A_i} + 1)$ of the module of differentials. We denote by $P_{\mu_{A_1}, \dots, \mu_{A_s}}$ the G_p -representation

$$k[G_p]/((\sigma_1 - 1)^{\mu_{A_1}+1}, \dots, (\sigma_s - 1)^{\mu_{A_s}+1})$$

where $\mu_{A_i} \in \{0, \dots, p^{t_i} - 1\}$, for each $i \in \{1, \dots, s\}$. For $\mu_{K_i} \in \{0, 1, \dots, n_i - 1\}$, we let V be a one-dimensional vector space over k . We define $D_{i,\mu_{K_i}}$ to be the representation on V such that the Galois action on any $v \in V$ is given by $\sigma^j(v) = \xi_i^{\mu_{K_i}j} v$, for all $j = 0, 1, \dots, n_i - 1$. Setting $\mu = (\mu_{A_1}, \dots, \mu_{A_s}, \mu_{K_1}, \dots, \mu_{K_r})$, we also denote

$$\Delta_{\mu} = P_{\mu_{A_1}, \dots, \mu_{A_s}} \otimes D_{1,\mu_{K_1}} \otimes \cdots \otimes D_{r,\mu_{K_s}}.$$

Then we have the $k[G]$ -isomorphism

$$U_{\mu,\nu} \simeq k[G]w_{\mu,\nu} \simeq \Delta_{\mu}.$$

Proof. $U_{\mu,\nu}$ is a $k[G]$ -module. Particularly, for any

$$h = (h_{A_1}, \dots, h_{A_s}, h_{K_1}, \dots, h_{K_r}) \in \prod_{i=1}^s \{0, \dots, p^{t_i} - 1\} \times \prod_{i=1}^r \{0, \dots, n_i - 1\},$$

we have

$$\sigma_{A_1}^{h_{A_1}} \cdots \sigma_{A_s}^{h_{A_s}} \sigma_{K_1}^{h_{K_1}} \cdots \sigma_{K_s}^{h_{K_s}}(\theta_{\mu'}^{\mu,\nu}) = \sum_{l=0}^{t^{\mu'} - t^{\mu}} B_{\mu,\mu',l} \sigma_{A_1}^{h_{A_1}} \cdots \sigma_{A_s}^{h_{A_s}} \sigma_{K_1}^{h_{K_1}} \cdots \sigma_{K_s}^{h_{K_s}}(w_{\mu',\nu+l})$$

$$= \sum_{l=0}^{t^{\mu'}-t^{\mu}} B_{\mu,\mu',l} \sum_{\mu'' \leq \mu'} c_{\mu',\mu'',h} \sum_{k=0}^{t^{\mu''}-t^{\mu'}} B_{\mu',\mu'',k} w_{\mu'',\nu+l+k}.$$

We recall that

$$h_{\mu,\mu'}(x) = \frac{g_{\mu'}(x)}{g_{\mu}(x)} = \sum_{l=0}^{t^{\mu'}-t^{\mu}} B_{\mu,\mu',l} x^l.$$

As a consequence, for all $\mu'' \leq \mu' \leq \mu$, we find that

$$h_{\mu,\mu''}(x) = h_{\mu,\mu'}(x) h_{\mu',\mu''}(x) = \sum_{l=0}^{t^{\mu''}-t^{\mu}} B_{\mu,\mu'',l} x^l = \sum_{l=0}^{t^{\mu''}-t^{\mu}} \sum_{e+f=l} B_{\mu',\mu'',e} B_{\mu,\mu',f} x^l.$$

Thus,

$$\sigma_1 \cdots \sigma_r(\theta_{\mu'}^{\mu,\nu}) = \sum_{\mu'' \leq \mu'} c_{\mu',\mu'',h} \sum_{l=0}^{t^{\mu''}-t^{\mu}} B_{\mu,\mu'',l} w_{\mu'',\nu+l} = \sum_{\mu'' \leq \mu'} c_{\mu',\mu'',h} \theta_{\mu''}^{\mu,\nu}$$

By the previous Lemma, $k[G]w_{\mu,\nu} \subseteq U_{\mu,\nu}$. Moreover, from Lemma 6.3, we obtain the $k[G]$ -epimorphism

$$\Delta_{\mu} \rightarrow k[G]w_{\mu,\nu} \subseteq U_{\mu,\nu}.$$

As the dimensions of Δ_{μ} and $U_{\mu,\nu}$ are the same, we find the desired isomorphisms. \square

Remark 6.6. If we take $s = 1$ and $r = 1$, that is, that the Galois group of L/K is cyclic, then the $k[G]$ -module Δ_{μ} is indecomposable [13]. We denote

$$\Omega_L^{\mu_{A_1}} = \{w \in \Omega_L \mid (\sigma_{A_1} - 1)^{\mu_{A_1}} w = 0\}.$$

Using the basis \mathfrak{B}_L , one easily obtains

$$\dim_k(\Omega_L^{\mu_{A_1}+1}/\Omega_L^{\mu_{A_1}}) = \sum_{\mu_{K_1} \in \{0, \dots, n_1\}} (t^{(\mu_{A_1}, \mu_{K_1})} - 1).$$

One can show that

$$\dim_k(\Omega_L^{\mu_{A_1}+1}/\Omega_L^{\mu_{A_1}}) = g_{L^{G_p}} - 1 + t^{\mu_{A_1}},$$

where $t^{\mu_{A_1}}$ is defined in analogy to (8) so that $\lambda'_{\mathcal{P}}^{\mu}$ and $\rho'_{\mathcal{P}}^{\mu}$ satisfy

$$(9) \quad e_{\mathfrak{P}|\mathfrak{p}_{Ku}} \lambda'_{\mathcal{P}}^{\mu} + \rho'_{\mathcal{P}}^{\mu} = \sum_{i=r}^t \sum_{j=0}^{u_i} e(\mathfrak{P}|\mathfrak{p}_{A_{i,j}})((p-1-\mu_{A_{i,j}})(-v_{p,A_{i,j}}) + (p-1)),$$

where $0 \leq \rho'_{\mathcal{P}}^{\mu} \leq e_{\mathcal{P}} - 1$. Also, we let

$$t^{\mu} = \sum_{\mathcal{P} \in \mathbb{P}_{L^{G_p}}} d_{\mathcal{P}} \lambda'_{\mathcal{P}}^{\mu}.$$

In this way, we recover Theorem 1 of [29]. Proceeding similarly, one may obtain a description of the Galois module structure of Ω_L in a more general setting than cyclic extensions, provided that the conditions of Theorem 4.1 are satisfied.

7 Concluding remarks

The following are some examples which demonstrate that there is much to understand about when Theorem 4.1 may be applied to describe the structure of the full automorphism group of a function field over its constant field.

Example 7.1 (*Fermat curves*). Let $k = \mathbb{F}_q$ ($q = p^h$) and K_n/k ($(n, p) = 1$) the function field defined by the equation

$$x^n + y^n = 1.$$

We let ζ_n denote a primitive n th root of unity, and we suppose that $n \mid (q - 1)$. This function field admits two types of automorphisms over \mathbb{F}_q . The first type of automorphism sends $x \rightarrow \zeta_n^a x$ and $y \rightarrow \zeta_n^b y$ ($a, b \in \mathbb{Z}/n\mathbb{Z}$). We let R_n denote the subgroup consisting of maps of this type. The second type of automorphism consists of two maps: one, which we call S , sending $x \rightarrow -\frac{y}{x}$ and $y \rightarrow \frac{1}{x}$, and the second, which we call T , sending $x \rightarrow \frac{1}{x}$ and $y \rightarrow -\frac{y}{x}$. We have $S^3 = T^2 = 1$ and $T^{-1}ST = S^{-1}$. We let H denote the group generated by T and S . As noted by Leopoldt [16], if $n - 1$ is not a power of p , then the automorphism group of the Fermat curve is given by $G_n = R_n H$. We have $R_n \trianglelefteq G_n$, so that the group G_n is the semidirect product of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with a dihedral group of order 6. As given by Lang [15], the space of holomorphic differentials of this curve is generated over k by the Boseck basis (the same as that given by Boseck [3]) consisting of elements

$$\omega_{r,s} = x^{r-1} y^{s-n} dx$$

where $r, s \geq 1$ and $r + s \leq n - 1$. This is slightly different from the basis given in Lemma 3.6. The basis $\{\omega_{r,s}\}$ provides a representation of R_n via action on the term $x^r y^s$ within $\omega_{r,s}$. The fixed field of R_n is equal to $K^{R_n} = \mathbb{F}_q(x^n) = \mathbb{F}_q(y^n)$. Via the generating equation of the Kummer generator y of K_n over $\mathbb{F}_q(x)$, the place of $\mathbb{F}_q(x)$ at infinity for x is unramified in K_n . Thus, the conditions of Theorem 4.1 are satisfied for $K_n/\mathbb{F}_q(x)$. By Lüroth's theorem [22], the fixed field F of the automorphism group of K_n over \mathbb{F}_q is rational. We do not know if it is possible to obtain the basis of Theorem 3.1 for the full extension K_n/F .

Example 7.2 (*Artin-Mumford curves*). If the field of constants $k = \overline{\mathbb{F}}_p$, the Artin-Mumford curve \mathcal{M}_c for a given $c \in k$ is defined as

$$(x^p - x)(y^p - y) = c.$$

The group of automorphisms of this curve over $\overline{\mathbb{F}}_p$ forms a semidirect product of a direct product $C_p \times C_p$ of two cyclic groups, each of order p , by the dihedral group D_{p-1} [28]. The place at infinity of $k(x)$ is unramified in \mathcal{M}_c , as can be seen by examining the generating equation of the Artin-Schreier generator y of \mathcal{M}_c over $k(x)$. Thus, the conditions of Theorem 4.1 are satisfied for \mathcal{M}_c over $k(x)$. As with the Fermat curve in Example 7.1, the fixed field F of the automorphism group of \mathcal{M}_c is rational [2]. However, we do not know if it is possible to obtain the basis of Theorem 4.1 for the extension \mathcal{M}_c/F .

Remark 7.1. Given a Galois extension of function fields $K/k(x)$ with field of constants k for which $\text{Gal}(K/k(x)) \trianglelefteq \text{Aut}_k(K)$ and fixed field F of $\text{Aut}_k(K)$, we may examine the Galois group of $k(x)/F$. If k is algebraically closed, the possible ramification behaviors and group structures

of $k(x)/F$ are described completely in [28]. This allows for a description of the Boseck basis and representation of $\text{Aut}_k(K)$ in terms of the extension K/F via Theorem 4.1 if the necessary criteria are satisfied.

Example 7.3 (*Hermitian curves*). Suppose that the field of constants $k = \mathbb{F}_{q^2}$. Let $K = k(x, y)/k$ be the Hermitian curve, defined by the equation

$$y^q + y = x^{q+1}.$$

The automorphism group of K/k is large (i.e., it does not satisfy the Hurwitz bound $|\text{Aut}_k(L)| \leq 84(g-1)$) and isomorphic to $PGU(3, q)$, of order $(q^3 + 1)(q^2 - 1)q^3$. As with the Fermat curve, one may define the basis of holomorphic differentials in terms of the Kummer extension $K/k(y)$. However, as the automorphism group of the Hermitian curve is in general not solvable, the result of Theorem 4.1 thus may not be applied in this case.

This work raised some additional questions for function fields in characteristic $p > 0$ (where the constant field is not assumed to be algebraically closed). These questions are related to when one may construct a basis as in Theorem 4.1 and emanate from connections between Galois and ramification theory.

- (i) In what generality does a global standard form exist? (See §5.)
- (ii) When would a tower of Kummer and Artin-Schreier extensions form a Galois extension?
- (iii) Given a non-Galois tower L/K , when does it occur that the index of ramification, inertia degree, and differential exponent are independent of the choice of place of L above a given place of K , for all places of K ?
- (iv) It would be interesting to find a non-abelian Galois tower for which Theorem 4.1 is valid.
- (v) When would there exist a subgroup of the automorphism group of a function field with a rational fixed field, such that the place at infinity is unramified?
- (vi) Is it possible to find an explicit basis of holomorphic differentials for a tower in the presence of an unramified step? (See §3.)
- (vii) Is it possible to construct a Boseck basis - in a tower or otherwise - when there does not exist an unramified place of degree one? (Even for cyclic automorphism groups, this would be interesting to know; this never occurs when the constant field k is algebraically closed, but it is quite common when k is finite.)
- (viii) In [23], the automorphism group G over the rational field is isomorphic to \mathbb{F}_q ($q = p^n$), and only $q - 1$ isomorphism classes of indecomposable $k[G]$ -modules are needed to completely describe the structure of Ω_L as a $k[G]$ -module. The aforementioned paper is the first to treat this problem when the automorphism group has non-cyclic p -part. The class of p -elementary abelian extensions addressed therein is hard to precisely describe, as they require the existence of a global standard form for their field generator, and even a local standard form is in general not guaranteed to exist (§5). These conditions also imply, in part, that all ramification is full, and require the constant field to contain \mathbb{F}_q . Some natural questions arise: Can the same $q - 1$ classes of indecomposable modules fully describe the $k[G]$ -module structure decomposition of Ω_L for p -elementary abelian extensions of a rational field with only partial ramification, or when the constant field does not contain

\mathbb{F}_q ? To what extent do these indecomposable modules give information about L ? For the basis of Theorem 4.1, we know the Galois action precisely (§6), but even if we were to find a $k[G]$ -submodule of Ω_L , it would be very difficult to determine whether or not it is indecomposable. When G is only assumed to be abelian, we leave it as an open question as to whether the Galois action given in §6 on the basis of Theorem 4.1 allows one to express Ω_L in terms of indecomposable $k[G]$ -modules. Any progress on these questions would constitute significant advances in Galois and representation theory in characteristic p .

Acknowledgements

We would like to thank the anonymous referees for their various insightful questions and comments.

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